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On the Radiation Field of a Perfectly Conducting Base Insulated Cylindrical Antenna over a Perfectly Conducting Plane Earth, and the Calculation of Radiation Resistance and Reactance

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XI—On the Radiation Field of a Perfectly Conducting Base Insulated Cylindrical Antenna Over a Perfectly Conducting Plane Earth, and the Calculation of Radiation Resistance and Reactance

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1—Introduction

The art of radio-broadcasting has in recent years directed attention to the radiation characteristics of the vertical antenna in circumstances where the height is comparable to the wave-length, λ. Existing theory generally assumes a sinusoidal distribution of current along the antenna. By integrating Hertz's expression for the radiation field of a current element it is indeed possible to obtain expressions for the electric and magnetic vectors of the entire antenna, and ultimately, by making use of Poynting's theorem over a very large hemisphere, to arrive at the well-known expression for the radiation resistance. Such an antenna, known as a *sine-wave antenna*, gives results in tolerably good agreement with observation as long as l/λ does not exceed $\frac{1}{4}$. This theory, admittedly imperfect, predicts sharp nodes of current along the antenna when $l/\lambda > \frac{1}{2}$, with the result that the radiation resistance calculated with respect to the current at the base of the antenna tends to infinite values at $l/\lambda = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, etc. Then again the theory of the sine-wave antenna ignores a prescribed boundary condition over the surface of the conductor. In the case of a perfectly conducting antenna this requires that the component E_z of the electric field along the antenna should be zero over a cylinder of radius $\rho = a$. The sine-wave solution, however, gives rise to two components of E_z in phase quadrature, neither of which is zero along the antenna, while one of them tends to infinite values at the foot and at the top of the antenna.*

Recent careful measurements of current distribution along an antenna of uniform cross-section have recently been carried out by Professor P. O. Pedersen (1935) at Copenhagen. His results for $l/\lambda = 0\cdot581$ show that the root mean square of current is approximately sinusoidal, but shows a minimum instead of a zero value at the node anticipated by the theory of the sine-wave antenna.

Many attempts have been made to improve the theory of the antenna by the use of the familiar transmission-line formulae. Results obtained by this means are admittedly imperfect and incorrect in principle, since the ideas of self-induction and capacity per unit length of conductor are based on the existence of electrostatic and magnetic potentials. When the dimensions of the antenna are comparable with

* This is apparent from the first term of equation (11) as r_1 and r tend to zero.

the wave-length, such potentials must be replaced by wave-potentials, and in these circumstances it is difficult to estimate the accuracy of line-transmission calculations.

It might be thought that from the measured current distribution it would be possible by integration of the field due to each current element, to arrive at expressions for the electric and magnetic vectors, and hence at an evaluation of radiation resistance. It turns out, however, on examination of the problem according to the electromagnetic field equations, that prescribed boundary conditions along the antenna lead to two components of antenna-current in phase quadrature, the relative amplitudes of which vary with distance from the earth. Measurements, however, are only capable of giving the root mean square of these components. Unless the two components can be resolved separately by more refined observations, the method of integrating the root mean square of current is theoretically incorrect and cannot be expected to give more than somewhat rough results.

It will be seen from this brief review of the situation that a solution of the antenna problem in terms of the electromagnetic field equations only, with due regard to the boundary conditions along the conductor, is desirable. The solution of the problem as developed in the following sections is found to depend on an integral equation. A first approximation of sufficient accuracy to be of interest as a practical solution, is obtained by analytical methods. Second and higher approximations, however, lead to refractory integrals whose evaluation is only possible by quadratures or methods of machine integration.

Although the present paper deals only with a perfectly conducting base insulated cylindrical antenna over a flat perfectly conducting earth, it is quite possible to introduce into the fundamental integral equation appropriate terms representing the effect of the ohmic resistance of the conductor and the reaction of an imperfectly conducting dielectric earth. In addition, the effect of a variable cross-section may be studied with little additional labour, a point of some importance in view of the mechanical structure of modern broadcasting towers.

2—Formulation of the Field Equations

When a radiation field is symmetrical with respect to the axis of z, it is completely specified in terms of the resultant magnetic field H. In the antenna problem the lines of force are circles about the axis of symmetry. If ρ is the distance of a point from the axis, it is convenient to make use of the function ψ defined by the relation

$$\psi = \rho H, \qquad (1)$$

so that ψ is proportional to the work done in taking unit magnetic pole around a circle of radius ρ.

In solid polar coordinates, Maxwell's equations may be written

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\psi}{\partial\mu^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \qquad (2)$$

$$\dot{E}_r = -\frac{c}{r^2}\frac{\partial\psi}{\partial\mu}, \qquad \dot{E}_\theta = -\frac{c}{r\sin\theta}\frac{\partial\psi}{\partial r}, \qquad H = \frac{\psi}{r\sin\theta}, \qquad (3)$$

where, as usual, $\mu = \cos\ \theta$ and E_r, E_θ are the components of the electric vector along and perpendicular to r respectively, while the dots above the symbols denote differentiation with respect to the time. The electric field is measured in absolute electrostatic units, the magnetic field in electromagnetic units, while c, as usual, denotes the velocity of propagation of electromagnetic waves *in vacuo*.

If the solution of (2) is expressed in cylindrical coordinates, the components of the electrical field consistent with (3) are

$$\dot{E}_\rho = -\frac{c}{\rho}\frac{\partial\psi}{\partial z}, \qquad \dot{E}_z = \frac{c}{\rho}\frac{\partial\psi}{\partial\rho}. \qquad H = \frac{\psi}{\rho}. \quad \ldots \ldots \quad (4)$$

At the surface of an elongated conductor, such as an antenna of radius a, inside of which there is no appreciable radiation field, we have immediately by Ampere's theorem

$$2, w\ (z) = (\psi)_{\rho=a}, \quad \ldots \ldots \quad (5)$$

where $w\ (z)$ is the antenna current.

If we introduce the time-factor $e^{i\omega t}$ into equation (2) and write $\kappa = w/c$, we make use of the particular solution

$$\psi = \text{constant} \times e^{-i\kappa r}, \quad \ldots \ldots \quad (6)$$

which obviously represents a divergent wave when r is always measured in the positive sense. It is seen from (1) that we may superimpose any number of such solutions referred to poles on the z-axis.

We now denote by r', r'_1, and r'_2 the distances of a point P (fig. 1) from the origin O, and the points $z = \pm\ \lambda$ on the z-axis. It then appears that the expression

$$\psi = A'\ \{e^{-i\kappa r'_1} + e^{-i\kappa r'_2} - 2\cos\kappa\lambda e^{-i\kappa r'}\} \quad \ldots \ldots \quad (7)$$

represents a divergent wave field satisfying Maxwell's equations (2) and (3). When r'_1, r'_2, and r are expressed in cylindrical coordinates (ρ, z) it is easily seen from the first of equations (4) that $E_\rho = 0$ over the plane $z = 0$ taken as a perfectly conducting earth in the antenna problem. If we place P at any point on the positive part of the z-axis such that $z > \lambda$ it is easily seen that $\psi = 0$, while if $z < \lambda$ we find that

$$(\psi)_{\rho=0} = -\ 2iA'\sin\kappa\ (\lambda - z). \quad \ldots \ldots \quad (8)$$

It thus follows from (5) that (7) is the solution appropriate to a sine-wave antenna of height l when $a \to 0$, and in fact on making use of equations (3) all the properties of the sine-wave antenna may be deduced from it. The results thus obtained are identical with those derived from the integration of Hertz's expression for the radiation field due to a current element assuming the sinusoidal distribution implied in equation (8).

It is now readily seen that the expression

$$\psi = A\ \{e^{-i\kappa r_1} + e^{-i\kappa r_2} - 2\cos\kappa l e^{-i\kappa r}\} + \int_e^l \phi\ (\lambda)\ \{e^{-i\kappa r'_1} + e^{-i\kappa r'_2} - 2\cos\kappa\lambda c^{-i\kappa r}\}\ d\lambda \quad (9)$$

represents a divergent wave solution of the field equations (2) and (3) appropriate to a cylindrical antenna of finite radius a very small compared with the wave-length. The first term, in which r, r_1, and r_2 are distances measured in the positive sense from the origin, the upper end of the antenna of height l and its " image " in the plane $z = 0$, corresponds to a sine-wave antenna. Without loss of generality A may be taken to be a positive constant.

The second term in (9) represents a continuous distribution of solutions of the type (6) or (7) along the z-axis between the base of the antenna at a height ε above earth and the upper end at $z = l$. $\phi(\lambda)$ is a distribution function to be determined from the boundary conditions along the antenna. Like the solution (7), it is readily

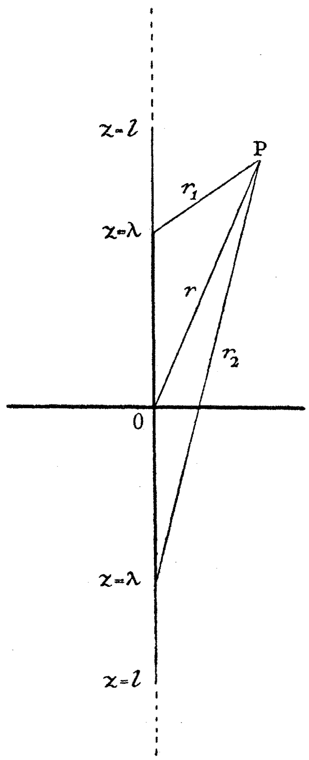

Fig. 1.

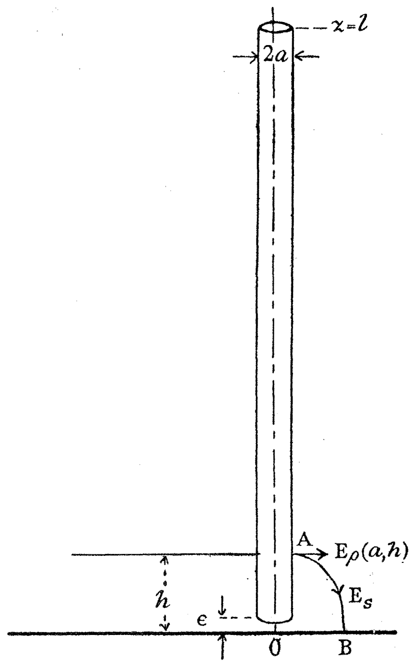

Fig. 2.

seen that the more general solution (9) gives $E_\rho = 0$ over the plane $z = 0$, while according to (5) and (8) we have

$$\left.\begin{aligned} w(z) &= -Ai\sin\kappa(l-z) - i\int_z^l \phi(\lambda)\sin\kappa(\lambda - z)\,d\lambda, \qquad &(0 < z < l) \\ w(z) &= 0 &(l < z < \infty) \end{aligned}\right\} . \qquad (10)$$

According to (4), we easily find at any point in space

$$-E_z = A\left\{\frac{e^{-i\kappa r_1}}{r_1} + \frac{e^{-i\kappa r_2}}{r_2} - 2\cos\kappa l\,\frac{e^{-i\kappa r}}{r}\right\} + \int_\varepsilon^l \phi(\lambda)\left\{\frac{e^{-i\kappa r_1'}}{r'_1} + \frac{e^{-i\kappa r'_2}}{r'_2} - 2\cos\kappa\lambda\,\frac{e^{-i\kappa r'}}{r'}\right\}d\lambda. \qquad \ldots\ldots (11)$$

In this paper we confine outselves to the perfectly conducting base insulated antenna of constant radius a. Over the surface of the antenna $E_z = 0$, so that (11) gives for the determination of $\phi(\lambda)$ the integral equation

$$-(E_z)_{\rho=a} = A\left[\frac{e^{-i\kappa\{(l-z)^2+a^2\}^{\frac{1}{2}}}}{\{(l-z)^2+a^2\}^{\frac{1}{2}}} + \frac{e^{-i\kappa\{(l+z)^2+a^2\}^{\frac{1}{2}}}}{\{(l+z)^2+a^2\}^{\frac{1}{2}}} - 2\cos\kappa l\,\frac{e^{-i\kappa\{z^2+a^2\}^{\frac{1}{2}}}}{\{z^2+a^2\}^{\frac{1}{2}}}\right]$$
$$+\int_\varepsilon^l \phi(\lambda)\left[\frac{e^{-i\kappa\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}}}{\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}} + \frac{e^{-i\kappa\{(\lambda+z)^2+a^2\}^{\frac{1}{2}}}}{\{(\lambda+z)^2+a^2\}^{\frac{1}{2}}}\right.$$
$$\left. - 2\cos\kappa\lambda\,\frac{e^{-i\kappa\{z^2+a^2\}^{\frac{1}{2}}}}{\{z^2+a^2\}^{\frac{1}{2}}}\right] d\lambda = 0. \qquad (12)$$

In general $\phi(\lambda)$ is complex and equivalent to two functions which must be determined to make both the real and imaginary parts of E_z vanish along the antenna. When $\phi(\lambda)$ has been determined to satisfy this condition, all the boundary conditions of the problem are very nearly satisfied since we have chosen the solution (9) to make $E_\rho = 0$ over the plane $z = 0$ and $w(z) = 0$ in the interval $l < z < \infty$.

It is quite true that the solution does not make $E_\rho = 0$ over the flat ends of the antenna, but once $\phi(\lambda)$ is known it is possible to trace, at $z = l$ from $\rho = a$ to the axis, a surface of revolution over which the tangential component of the electric force is zero. The same remark applies to the lower end of the antenna at $z = \varepsilon$, so that the solution applies to a perfectly conducting antenna with somewhat arbitrarily rounded ends. But in the case of an elongated conductor in which a/l is small, the variation of antenna characteristics arising from such deviations from flat ends is negligible.

Similarly equation (10) for the distribution of antenna current is obtained by placing the point P (fig. 1) on the axis of z. When κa (= ratio of antenna circumference to wave-length) is very small, and a/l also small, there is no difficulty in showing that $(\psi)_{\rho=a}$ differs from $(\psi)_{\rho=0}$ by negligible quantities, so that according to (5) the current distribution calculated from (10) may be considered sufficiently accurate.

3—Approximate Solution of the Fundamental Integral Equation

In the following sections we replace ε by 0, *i.e.*, we take the insulating gap at the foot of the antenna to be so small that the disturbance of the radiation field due to the existence of such a gap is negligible compared with that contributed by the antenna as a whole.

The solution of the integral equation (12) by successive approximations depends on the properties of the integral

$$M(\lambda) = \int_0^l \frac{e^{-i\kappa\{(u-\lambda)^2+a^2\}^{\frac{1}{2}}}}{\{(u-\lambda)^2+a^2\}^{\frac{1}{2}}}\,du. \qquad (13)$$

It is a simple matter to prove, by writing $l - u$ for u, that

$$M(l-\lambda) = M(\lambda), \qquad (14)$$

and, in particular,

$$M(l) = M(0) = \int_0^l \frac{e^{-i\kappa(u^2+a^2)^{\frac{1}{2}}}}{(u^2+a^2)^{\frac{1}{2}}}\,du. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We seek, for $\phi(\lambda)$, a solution of the form

$$\phi(\lambda) = \frac{1}{M(\lambda)}\left[p\,\frac{e^{-i\kappa\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}}}{\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}} + q\,\frac{e^{-i\kappa\{(l+\lambda)^2+a^2\}^{\frac{1}{2}}}}{\{(l+\lambda)^2+a^2\}^{\frac{1}{2}}} + r\,\frac{e^{-i\kappa\{\lambda^2+a^2\}^{\frac{1}{2}}}}{\{\lambda^2+a^2\}^{\frac{1}{2}}}\right] + \chi(\lambda), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where p, q, and r are constants to be adjusted to suit the requirements of the problem and $\chi(\lambda)$ represents a second approximation to $\phi(\lambda)$.

On introducing $\phi(\lambda)$ into (12) we have to determine approximate values of nine integrals of which a typical one is

$$\int_0^l \frac{e^{-i\kappa\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}}}{\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}} \cdot \frac{e^{-i\kappa\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}}}{\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}} \cdot \frac{d\lambda}{M(\lambda)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

When κa is very small and a/l small, the above integral has two pronounced maxima in the interval $0 < \lambda < l$, one in the neighbourhood of $\lambda = l$, the other near $\lambda = z$.

Since $M(\lambda)$ is monotonic, varies slowly over the greater part of the range of integration, and is finite at $\lambda = 0$ and $\lambda = l$, while the exponential terms are likewise finite, it follows that the integral (17) depends principally on the values of the integrand in the neighbourhood of these maxima. To obtain an approximate value of the integral, we note that in the circumstances cited above

$$\frac{1}{\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}} \sim \frac{1}{\{(l-z)^2+a^2\}^{\frac{1}{2}}} \times \left[\frac{1}{\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}} + \frac{1}{\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}}\right], \quad (18)$$

a relation which becomes exact as $\kappa a \to 0$ and the positive values of $|l-\lambda|$, $|\lambda - z|$, and $[l - z]$ are taken to conform to the requirement that ψ represent a divergent wave.

The integral (17) is thus approximately equal to the sum of two integrals, one of which takes its value almost entirely in the neighbourhood of $\lambda = l$, the other in the neighbourhood of $\lambda = z$.

In the integral corresponding to the first term of (18) we may write $\lambda = l$ in the terms $1/M(\lambda)$ and $e^{-i\kappa\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}}$ of the integrand and its value is, approximately,

$$\frac{1}{M(l)}\,\frac{e^{-i\kappa\{(l-z)^2+a^2\}^{\frac{1}{2}}}}{\{(l-z)^2+a^2\}^{\frac{1}{2}}}\int_0^l \frac{e^{-i\kappa\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}}}{\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}}\,d\lambda = \frac{e^{-i\kappa\{(l-z)^2+a^2\}^{\frac{1}{2}}}}{\{(l-z)^2+a^2\}^{\frac{1}{2}}}, \quad . \quad (19)$$

since, according to (13), the integral in (19) is $M(l)$.

Similarly, in the integral corresponding to the second term of (18), we may write $\lambda = z$ in the terms $1/\mathrm{M}(\lambda)$ and $e^{-i\kappa\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}}$ of the integrand. The value of this integral is, approximately,

$$\frac{1}{\mathrm{M}(z)} \frac{e^{-i\kappa\{(l-z)^2+a^2\}^{\frac{1}{2}}}}{\{(l-z)^2+a^2\}^{\frac{1}{2}}} \int_0^l \frac{e^{-i\kappa\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}}}{\{(\lambda-z)^2+a^2\}^{\frac{1}{2}}}\, d\lambda = \frac{e^{-i\kappa\{(l-z)^2+a^2\}^{\frac{1}{2}}}}{\{(l-z)^2+a^2\}^{\frac{1}{2}}}, \quad . \quad (20)$$

since, according to (13), the integral in the above expression is M (z).

It thus follows that the approximate value of the integral (17) is

$$2\, \frac{e^{-i\kappa\{(l-z)^2+a^2\}^{\frac{1}{2}}}}{\{(l-z)^2+a^2\}^{\frac{1}{2}}}.$$

By proceeding in this way with each of the nine integrals arising from the substitution of the first three terms of (16) in the integral equation (12), we have

$$\int_0^l \phi(\lambda)\, [\ldots]\, d\lambda \sim 2p\, \frac{e^{-i\kappa\{(l-z)^2+a^2\}^{\frac{1}{2}}}}{\{(l-z)^2+a^2\}^{\frac{1}{2}}} + (p+q)\, \frac{e^{-i\kappa\{(l+z)^2+a^2\}^{\frac{1}{2}}}}{\{(l+z)^2+a^2\}^{\frac{1}{2}}}$$
$$+ (r - 2p\cos\kappa l)\, \frac{e^{-i\kappa\{z^2+a^2\}^{\frac{1}{2}}}}{\{z^2+a^2\}^{\frac{1}{2}}} + \int_0^l \chi(\lambda)\, [\ldots]\, d\lambda. \quad (21)$$

If we now choose the constants p, q, r so that

$$p + q = 2p = -\mathrm{A} \quad \text{and} \quad r - 2p\cos\kappa l = -4p\cos\kappa l, \quad . \quad . \quad (22)$$

we see that the integral equation (12) is approximately satisfied and $\chi(\lambda)$ is to be determined as a second approximation from an integral equation of the type (12) in which the first term is replaced by the difference between the exact and approximate values of the nine integrals of the type (17).

On replacing p, q, r in (16) by the values determined by equations (22) and denoting $\alpha = \kappa l$, we have as a first approximation to the solution of the integral equation

$$\phi(\lambda) = -\frac{\mathrm{A}}{2\mathrm{M}(\lambda)} \left[\frac{e^{-i\kappa\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}}}{\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}} + \frac{e^{-i\kappa\{(l+\lambda)^2+a^2\}^{\frac{1}{2}}}}{\{(l+\lambda)^2+a^2\}^{\frac{1}{2}}} \right.$$
$$\left. - 2\cos\alpha\, \frac{e^{-i\kappa\{\lambda^2+a^2\}^{\frac{1}{2}}}}{\{\lambda^2+a^2\}^{\frac{1}{2}}} \right] + \chi(\lambda). \quad (23)$$

To obtain $\chi(\lambda)$ it is necessary to substitute $\phi(\lambda)$ in (12) and determine the exact values of the integrals, which, unfortunately, can only be evaluated by quadratures or methods of machine integration. It is possible, however, to show that $\chi(\lambda)$ depends on $[2\mathrm{M}(\lambda)]^{-2}$. It will be shown in a subsequent section that M (λ) depends principally on $2\log(2l/a)$, so that the effect of retaining the first three terms of (10) as a first approximation to $\phi(\lambda)$ is to neglect terms of the order $[2\log(2l/a)^{-2}]$ in the calculated characteristics of the antenna. To this order of approximation it is possible to examine the radiation field of a perfectly conducting thin antenna of uniform cross-section and to ascertain the corrections to be made to the theory of the sine-wave antenna whose properties are described by the first term of equation (9).

4—Mathematical Functions Associated with Antenna Characteristics and their Numerical Tabulation

In connexion with the sine-wave antenna it is convenient to make use of the functions $S_1(\alpha)$ and $S_2(\alpha)$ defined by the integrals

$$S_1(\alpha) = \int_0^\alpha (1 - \cos x)\,\frac{dx}{x} \qquad S_2(\alpha) = \int_0^\alpha \sin x\,\frac{dx}{x}. \quad . \quad . \quad . \quad (24)^*$$

These integrals are connected with the well-known sine- and cosine-integrals

$$Si(\alpha) = \int_0^\alpha \sin x\,\frac{dx}{x} \qquad \text{and} \qquad Ci(\alpha) = -\int_\alpha^\infty \cos x\,\frac{dx}{x} \quad . \quad . \quad . \quad (25)$$

by the relations

$$S_2(\alpha) = Si(\alpha) \qquad S_1(\alpha) = r + \log \alpha - Ci(\alpha) \quad . \quad . \quad . \quad (26)$$

where r is Euler's constant, $r = 0\cdot5772157$.

The calculation of the properties of the perfectly conducting antenna requires the introduction of the functions

$$F(\alpha) = \int_0^{\frac{1}{2}\pi} \{\cos(\alpha\cos\theta) - \cos\alpha\}\,\frac{d\theta}{\sin\theta}, \qquad G(\alpha) = \frac{d}{d\alpha}F(\alpha). \quad . \quad (27)$$

We easily find that $F(\alpha)$ satisfies the differential equation

$$\frac{d^2F}{d\alpha^2} + F = \frac{\sin\alpha}{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

so that from a well-known theorem in differential equations

$$F(\alpha) = \int_0^\alpha \frac{\sin u}{u}\sin(\alpha - u)\,du \quad \text{and} \quad G(\alpha) = \int_0^\alpha \frac{\sin u}{u}\cos(\alpha - u)\,du. \quad (29)$$

Expansions of (27) and (29) near $\alpha = 0$ indicate that the constants A and B in the general solution, $A\sin\alpha + B\cos\alpha$, of (28) are both zero.

The following expansions are easily obtained from (28):

$$\left.\begin{aligned} F(\alpha) &= \frac{\alpha^2}{2!} - \frac{\alpha^4}{4!}(1 + \tfrac{1}{3}) + \frac{\alpha^6}{6!}(1 + \tfrac{1}{3} + \tfrac{1}{5}) - \dots \\ G(\alpha) &= \frac{\alpha}{1!} - \frac{\alpha^3}{3!}(1 + \tfrac{1}{3}) + \frac{\alpha^5}{5!}(1 + \tfrac{1}{3} + \tfrac{1}{5}) - \dots \end{aligned}\right\} \quad . \quad . \quad . \quad (30)$$

* Excellent tables of the functions $S_1(\alpha)$ and $S_2(\alpha)$ to five places of decimals from $\alpha = 0$ to $\alpha = 25\cdot0$ at intervals of $0\cdot1$ have been computed by Professor P. O. Pedersen, making use of extensive tables of the sine- and cosine-integrals by Tani (1931). With his kind permission, Professor Pedersen's tables are reproduced as tables A and B in the Appendix to this paper.

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From (29) we at once obtain

$$\left.\begin{aligned} F(\alpha) &= -\tfrac{1}{2}\cos\alpha S_1(2\alpha) + \tfrac{1}{2}\sin\alpha\, S_2(2\alpha) \\ G(\alpha) &= \tfrac{1}{2}\sin\alpha S_1(2\alpha) + \tfrac{1}{2}\cos\alpha\, S_2(2\alpha) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (31)$$

If we write $\tan\beta = S_2(2\alpha)/S_1(2\alpha)$ we have

$$F(\alpha) = -\tfrac{1}{2}S_1(2\alpha)\sec\beta\cos(\alpha+\beta), \quad G(\alpha) = \tfrac{1}{2}S_1(2\alpha)\sec\beta\sin(\alpha+\beta) \quad (32)$$

convenient for the construction of Table C of these functions from Pedersen's tables.

When κa is very small we have the following approximate evaluation of integrals required in a later section:

$$\left.\begin{aligned} \int_0^\alpha \frac{\cos u}{(u^2+\kappa^2a^2)^{\frac{1}{2}}}\sin(\alpha-u)\,du &\sim \sin\alpha\sinh^{-1}\{\alpha/(\kappa a)\} - G(\alpha) \\ \int_0^\alpha \frac{\cos u}{(u^2+\kappa^2a^2)^{\frac{1}{2}}}\cos(\alpha-u)\,du &\sim \cos\alpha\sinh^{-1}\{\alpha/(\kappa a)\} + F(\alpha) \end{aligned}\right\}, \quad . \; . \quad (33)$$

terms of the order κa being neglected.

The evaluation of radiation resistance by the application of Poynting's theorem over a very large hemisphere leads to the integral

$$R(\alpha) = 2\int_0^{\frac{1}{2}\pi}\{\cos(\alpha\cos\theta) - \cos\alpha\}^2\frac{d\theta}{\sin\theta}. \quad . \; . \; . \; . \; . \quad (34)$$

Since

$$2\{\cos(\alpha\cos\theta) - \cos\alpha\}^2 = \cos(2\alpha\cos\theta) - \cos 2\alpha - 4\cos\alpha\{\cos(\alpha\cos\theta) - \cos\alpha\},$$

we have, from (27),

while

$$\left.\begin{aligned} R(\alpha) &= F(2\alpha) - 4\cos\alpha\, F(\alpha), \\ L(\alpha) &= G(2\alpha) - 4\cos\alpha\, G(\alpha) \end{aligned}\right\}, \quad . \; . \; . \; . \; . \; . \; . \quad (35)$$

is also required in subsequent calculations, in the form

$$\left.\begin{aligned} U(\alpha) &= \sin\alpha\, L(\alpha) - \cos\alpha\, R(\alpha) \\ V(\alpha) &= \cos\alpha\, L(\alpha) + \sin\alpha\, R(\alpha) \end{aligned}\right\}. \quad . \; . \; . \; . \; . \; . \; . \quad (36)$$

In terms of the functions S_1 and S_2 we have

$$\left.\begin{aligned} U(\alpha) &= \tfrac{1}{2}S_1(4\alpha)\cos\alpha - \tfrac{1}{2}S_2(4\alpha)\sin\alpha - 2\cos\alpha S_1(2\alpha) \\ V(\alpha) &= \tfrac{1}{2}S_1(4\alpha)\sin\alpha + \tfrac{1}{2}S_2(4\alpha)\cos\alpha - 2\cos\alpha S_2(2\alpha) \end{aligned}\right\} \quad . \; . \; . \quad (37)$$

On writing $\tan\eta = S_2(4\alpha)/S_1(4\alpha)$ we have

$$\left.\begin{aligned} U(\alpha) &= \tfrac{1}{2}S_1(4\alpha)\sec\eta\cos(\alpha+\eta) - 2\cos\alpha S_1(2\alpha) \\ V(\alpha) &= \tfrac{1}{2}S_1(4\alpha)\sec\eta\sin(\alpha+\eta) - 2\cos\alpha S_2(2\alpha) \end{aligned}\right\}, \quad . \; . \quad (38)$$

convenient for purposes of tabulation. In this manner the tables D of the appendix were computed.

If we now write $\tan \xi = U(\alpha)/V(\alpha)$ we readily find from (36) that

$$R(\alpha) = V(\alpha) \sec \xi \sin(\alpha - \xi), \quad L(\alpha) = V(\alpha) \sec \xi \cos(\alpha - \xi), \quad . \quad . \quad . \qquad (39)$$

by means of which these functions were computed.

For small values of α the following series expansions are useful :

$$\left.\begin{array}{ll} R(\alpha) = \frac{1}{3}\alpha^4 \{1 - \frac{1}{5}\alpha^2 + \ldots\} & L(\alpha) = -2\alpha\{1 - \frac{5}{9}\alpha^2 \ \ldots\} \\ U(\alpha) = -2\alpha^2\{1 - \frac{5}{9}\alpha^2 \ldots\} & V(\alpha) = -2\alpha\{1 - \frac{19}{18}\alpha^2 \ldots\} \end{array}\right\}. \qquad (40)$$

We are now able to evaluate the function $M(\lambda)$, defined by (13), which occurs in the denominator of the expression (23) for $\phi(\lambda)$. When κa is very small we may neglect $\kappa^2 a^2$ in the exponential in (13) provided the positive value of $|u - \lambda|$ is taken.

We easily find

$$\begin{aligned} M(\lambda) &= M'(\lambda) - iM''(\lambda) \\ &= \sinh^{-1}\{(l-\lambda)/a\} + \sinh^{-1}(\lambda/a) - \{S_1\{\kappa(l-\lambda)\} + S_1(\kappa\lambda)\} \\ &\qquad - i\{S_2\{\kappa(l-\lambda)\} + S_2(\kappa\lambda\}. \qquad (41) \end{aligned}$$

In fig. 3 are given two typical graphs of $M'(\lambda)$ and $M''(\lambda)$ for antennae of dimensions corresponding to $\log(2l/a) = 4 \cdot 5$ and $\log(2l/a) = 10 \cdot 58$ at $l/\lambda = 0 \cdot 581$. $M(\lambda)$ is seen to remain very nearly constant over the range $0 < \lambda < l$, a circumstance which makes it possible to evaluate approximately the integrals arising when $\phi(\lambda)$ is introduced into the expression (9) for ψ. Throughout the remainder of this paper we replace $1/M(\lambda)$ by $1/\overline{M}(\alpha)$, where $\overline{M}(\alpha)$ is the mean value of $M(\lambda)$ defined by

$$l\overline{M}(\alpha) = \int_0^l M(\lambda)\, d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

The error in so doing is estimated to be of the same order as the neglect of the second approximation for $\phi(\lambda)$, *i.e.*, terms of the order $[2 \log(2l/a)]^{-2}$.

We easily find, on writing $\sinh^{-1}(l/a) \sim \log(2l/a)$, that

$$\overline{M}(\alpha) = \overline{M}'(\alpha) - i\overline{M}''(\alpha) = |\overline{M}(\alpha)|\, e^{-im}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

where

$$\left.\begin{array}{l} \overline{M}'(\alpha) = 2[\log(2l/a) - \{S_1(\alpha) + \sin\alpha/\alpha\}] \\ \overline{M}''(\alpha) = 2[S_2(\alpha) - (1 - \cos\alpha)/\alpha] \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

In the following sections we omit the bars over the M's, with the understanding that mean values of $M(\lambda)$ are used throughout.

According to (43),

$$\tan m = M''(\alpha)/M'(\alpha) \qquad \text{and} \qquad |M(\alpha)| = M'(\alpha) \sec m. \quad . \quad . \qquad (45)$$

The imaginary part of M (α) is small compared with the real part, but still sufficiently large to be included in all calculations of antenna characteristics.

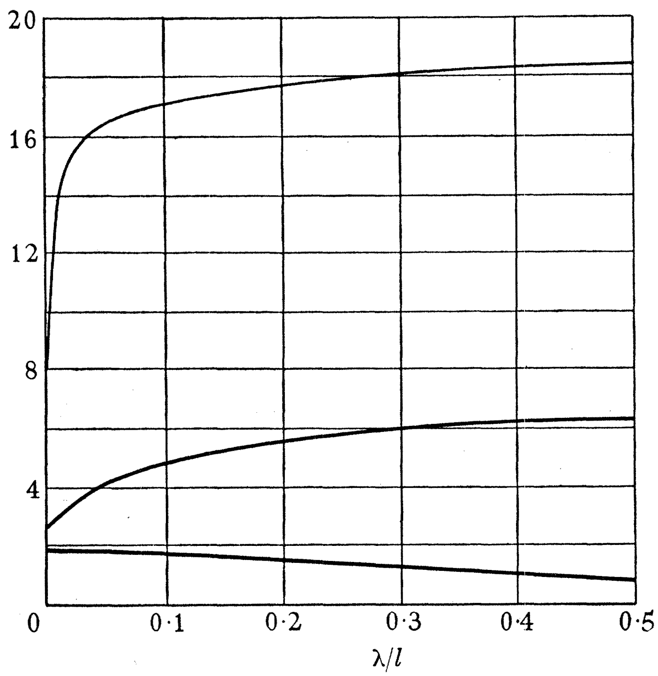

FIG. 3—*Graph of the Function* M (λ) = M′ (λ) − iM′ (λ).—The upper curve drawn with a thin line is the graph of M′ (λ) for log $(2l/a) = 10\cdot58$ and $\alpha = 3\cdot64$. The next curve represented by a thick line is the graph of M′ (λ) for log $(2l/a) = 4\cdot5$ and $\alpha = 3\cdot64$. The lowest curve represents M″ (λ) which is independent of log $(2l/a)$. All the curves are symmetrical with respect to the point $\lambda/l = 0\cdot5$. To estimate the error involved by replacing $1/\mathrm{M}(\lambda)$ by $1/\overline{\mathrm{M}}(\alpha)$ in integrals involving $\phi(\lambda)$ given by equation (20), we have determined by quadratures the mean value of $1/\mathrm{M}(\lambda)$ in the interval $0 < \lambda < l$ as defined by

$$l\,[1/\mathrm{M}(\lambda)] = \int_0^l d\lambda/\mathrm{M}(\lambda).$$

We find that for $\alpha = 3\cdot64$

log $(2l/a)$	$[1/\mathrm{M}(\lambda)]$	$1/\overline{\mathrm{M}}(\alpha)$
10·58	$0\cdot0552 + 0\cdot00812i$	$0\cdot0557 + 0\cdot00820i$
4·5	$0\cdot1542 + 0\cdot0736i$	$0\cdot1508 + 0\cdot0722i$,

from which it is evident that the approximation referred to is of the same order as that involved in the neglect of the second approximation for $\phi(\lambda)$.

It is evident that in evaluating integrals involving $\phi(\lambda)$ we may neglect κ^2a^2 in the exponential terms of (23) and with sufficient accuracy replace $1/\mathrm{M}(\lambda)$ by $e^{im}/|\mathrm{M}(\alpha)|$. We thus take, in the calculations of the following sections,

$$\phi(\lambda) \sim -\frac{Ae^{-im}}{2|\mathrm{M}(\alpha)|}\left[\frac{e^{-i\kappa(l-\lambda)}}{\{(l-\lambda)^2+a^2\}^{\frac{1}{2}}}+\frac{e^{-i\kappa(l+\lambda)}}{\{(l+\lambda)^2+a^2\}^{\frac{1}{2}}}-2\cos\alpha\,\frac{e^{-i\kappa\lambda}}{\{\lambda^2+a^2\}^{\frac{1}{2}}}\right], \quad (46)$$

and replace the sign $\sim$ of approximation by that of equality with the understanding that terms of the order $[2\log(2l/a)]^{-2}$ are neglected.

5—Current Distribution Along the Antenna

On introducing into equation (10) the approximate value of $\phi(\lambda)$ given by (46), we find on writing

$$\zeta = \kappa (l - z), \quad (47)$$

and carrying out the integrations by the use of the formulae of § 4 that

$$w(\zeta) = -iA\left[\sin\zeta - \frac{e^{im}}{2|M|}\{(w'_1 - iw''_1) + (w'_2 - iw''_2) + (w'_3 - iw''_3)\}\right], \quad (48)$$

where

$$\left.\begin{aligned} w'_1 &= \sin\zeta \sinh^{-1}\{\zeta/(\kappa a)\} - G(\zeta) \\ w'_2 &= \cos\zeta G(2\alpha) + \sin\zeta F(2\alpha) - G(2\alpha - \zeta) - \sin(2\alpha - \zeta)\log\{2\alpha/(2\alpha - \zeta)\} \\ w'_3 &= -2\cos\alpha\,[\cos\zeta G(\alpha) + \sin\xi F(\alpha) - G(\alpha - \zeta) \\ &\qquad - \sin(\alpha - \zeta)\{\log(2l/a) - \sinh^{-1}\{(\alpha - \zeta)/(\kappa a)\}\}] \end{aligned}\right\} \quad (49)$$

$$\left.\begin{aligned} w''_1 &= F(\zeta) \\ w''_2 &= -\cos\zeta F(2\alpha) + \sin\zeta G(2\alpha) + F(2\alpha - \zeta) \\ w''_3 &= -2\cos\alpha\{-\cos\zeta F(\alpha) + \sin\zeta G(\alpha) + F(\alpha - \zeta)\} \end{aligned}\right\}. \quad (50)$$

The terms denoted by the suffixes 1, 2, 3 correspond to the real and imaginary parts of the integrals arising from the three terms of $\phi(\lambda)$ in (46) taken in order, and are kept separate for convenience in computation.

To compute the real and imaginary components of (48), we write

$$S' = w'_1 + w'_2 + w'_3, \qquad S'' = w''_1 + w''_2 + w''_3, \qquad \text{and} \qquad \tan s = S''/S'. \quad (51)$$

Then, denoting

$$w(\zeta) = A\{w'(\zeta) - iw''(\zeta)\}, \quad (52)$$

we find that

$$w(\zeta) = -iA\{\sin\zeta - \tfrac{1}{2}|S|/|M| \,.\, e^{-i(s-m)}\},$$

from which it follows that

$$w'(\zeta) = \tfrac{1}{2}\frac{|S|}{|M|}\sin(s - m) \quad \text{and} \quad w''(\zeta) = \sin\zeta - \tfrac{1}{2}\frac{|S|}{|M|}\cos(s - m). \quad (53)$$

In fig. 4 are drawn graphs of the above current components for $\log(2l/a) = 4\cdot5$, $10\cdot58$, and ∞ at $\alpha = 2\pi l/\lambda = 3\cdot64$.

Comparison with Observations on the Copenhagen Antenna

The observations given by Professor Pedersen refer to measurements of current along an antenna for which $2a = 1\cdot50$ cm., $l = 148\cdot1$ metres at wave-length $255\cdot1$ metres. Thus $\log(2l/a) = 10\cdot58$ and $\alpha = 2\pi l/\lambda = 3\cdot64$. According to (44) and

(45), we find M $(\alpha) = 17 \cdot 56 - 2 \cdot 58i$ and from (45), $m = 8° - 22'$, $|M| = 17 \cdot 75$. From the entries of Table C in the Appendix large scale graphs of F (ζ) and G (ζ) were drawn, from which the w's in (49) and (50) were calculated by four-figure logarithms to three significant figures. These are entered in Table I below as well as the components w'' (ζ) and w' (ζ) computed from equation (53). In fig. 5, the

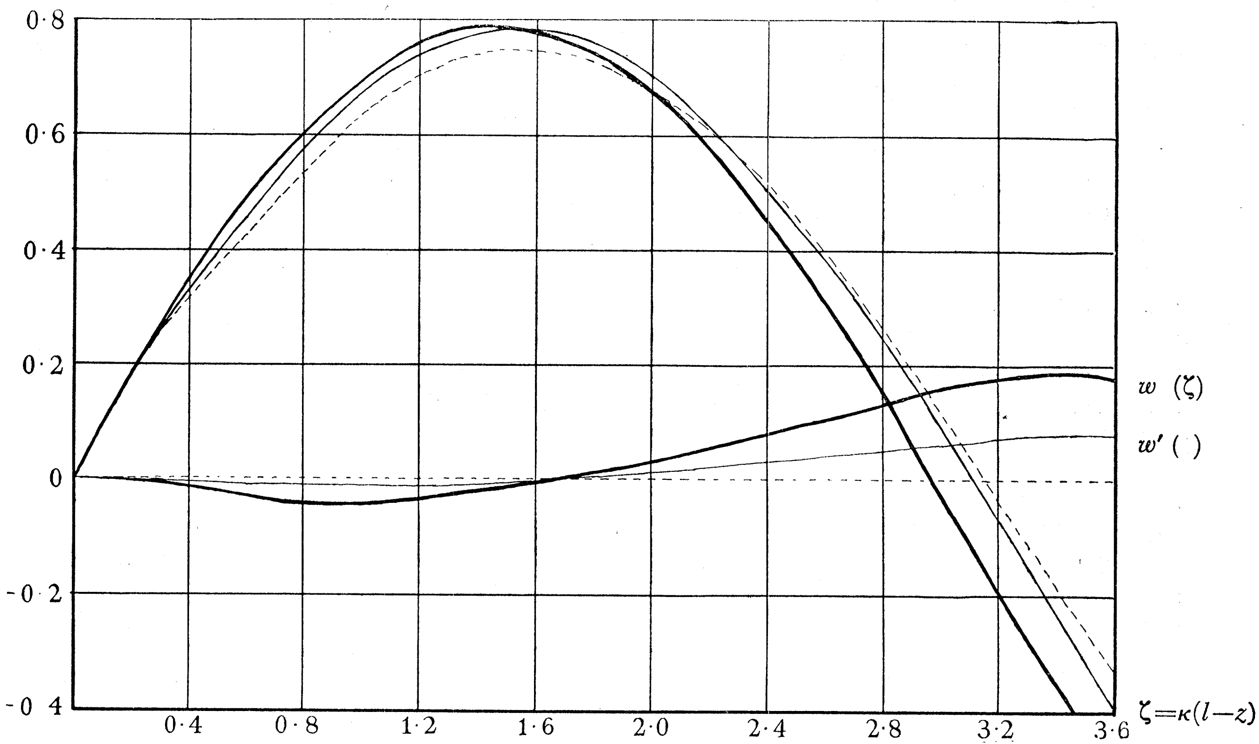

FIG. 4—*Current Components Along Perfectly Conducting Antenna*—The electromagnetic field solution of the perfectly conducting antenna requires that the electric field along it be zero. As a result, there are two components of current in phase quadrature, w'' (ζ) and w' (ζ) computed from equations (53) and entered in Table I. A graphical representation of the results is given above for $\alpha = 2\pi l/\lambda = 3 \cdot 64$ and log $(2l/a) = 4 \cdot 5$ (thick curve), log $(2l/a) = 10 \cdot 58$ (thin curve), and log $(2l/a) \rightarrow \infty$ (dotted curve). The last corresponds to the sine-wave antenna for which w' $(\zeta) = 0$. It is a difficult experimental problem to resolve these components at radio frequencies. Observations by means of search-coils only suffice to give the root-mean-square of the antenna current.

root-mean-square current, $|w\ (\zeta)|/\sqrt{2} = \mathrm{A}\ \{w''^2\ (\zeta) + w'^2\ (\zeta)\}^{\frac{1}{2}}/\sqrt{2}$ is plotted from the computations of Table II, and the curve corresponding to log $(2l/a) = 10 \cdot 58$ is seen to agree as well as could be desired with Pedersen's observations adjusted to make the current at the base of the antenna coincide with the theoretical value calculated for 1 kW radiation output.

On the same graph is plotted the theoretical current distribution for an antenna of dimensions log $(2l/a) = 4 \cdot 5$ for an antenna of the same length and radiation output of 1 kW at the same wave-length. It will be noticed that the minimum value of $|w\ (\zeta)|\ /\sqrt{2}$ is less sharply marked as log $(2l/a)$ decreases.

TABLE I

$$w(\zeta) = A\{w'(\zeta) - iw''(\zeta)\}, \qquad w''(\zeta) = \sin\zeta - \tfrac{1}{2}\frac{|S|}{|M|}\cos(s - m),$$

$$w'(\zeta) = \tfrac{1}{2}\frac{|S|}{|M|}\sin(s - m)$$

ζ	w''_1	w''_2	w''_3	w'_1	w'_2	w'_3	$w'(\zeta)$	$w''(\zeta)$	$\sin\zeta$
0·0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·0000	0·0000
0·3	0·045	0·004	−0·011	0·301	0·005	−0·018	−0·0076	0·2713	0·2954
0·6	0·173	0·016	−0·026	0·964	0·018	−0·084	−0·0234	0·4924	0·5645
0·9	0·370	0·033	−0·105	1·781	0·041	−0·192	−0·0368	0·6497	0·7833
1·2	0·610	0·049	−0·021	2·314	0·076	−0·316	−0·0238	0·7602	0·9320
1·5	0·867	0·055	+0·041	2·759	0·117	−0·550	−0·0121	0·7874	0·9975
1·8	1·105	0·048	0·160	2·966	0·161	−0·650	+0·00831	0·7399	0·9739
2·1	1·292	0·034	0·346	2·902	0·202	−0·801	0·0413	0·6296	0·8633
2·4	1·401	−0·011	0·604	2·554	0·235	−0·819	0·0785	0·4548	0·6756
2·5	1·416	−0·017	0·703	2·377	0·241	−0·805	0·0928	0·3857	0·5985
2·6	1·420	−0·046	0·812	2·173	0·246	−0·789	0·1054	0·3134	0·5155
2·7	1·410	−0·066	0·933	1·947	0·247	−0·745	0·1187	0·2355	0·4274
2·8	1·388	−0·089	1·042	1·699	0·252	−0·696	0·1305	0·1554	0·3350
2·9	1·354	−0·112	1·169	1·423	0·243	−0·668	0·1452	0·0751	0·2379
3·0	1·307	−0·136	1·301	1·139	0·235	−0·513	0·1547	−0·0135	0·1412
3·1	1·247	−0·161	1·434	0·840	0·226	−0·381	0·1648	−0·1016	0·0416
3·3	1·091	−0·212	1·703	0·201	0·193	−0·023	0·1807	−0·2824	−0·1571
3.5	0·890	−0·259	1·972	−0·454	0·144	+0·549	0·1870	−0·4633	−0·3507
3·6	0·774	−0·286	2·106	−0·780	0·120	1·247	0·1739	−0·5811	−0·4426

Current distribution along antenna for $\log(2l/a) = 4\cdot5, \alpha = 3\cdot64, A = 1, |M| = 5\cdot98, m^\circ = 25^\circ - 47'$, computed from formulae (49), (50), (51), (52), and (53). The variable $\zeta = \kappa(l - z)$ is proportional to the distance from the upper end of the antenna.

6—CALCULATION OF DISTANT RADIATION FIELD

On referring to formula (9) and fig. 1 we notice that at very great distances from the antenna we may write approximately

$$r_1 \sim r - l\cos\theta,\ r_2 \sim r + l\cos\theta,\ r'_1 \sim r - \lambda\cos\theta,\ r'_2 \sim r + \lambda\cos\theta,$$

so that we have

$$\psi \sim 2Ae^{-i\kappa r}\{\cos(\alpha\cos\theta) - \cos\alpha\} + 2e^{-i\kappa r}\int_0^l\{\cos(\kappa\lambda\cos\theta) - \cos(\kappa\lambda)\}\,\phi(\lambda)\,d\lambda. \quad (54)$$

We now write

$$\phi(\lambda) = \phi'(\lambda) - i\,\phi''(\lambda), \quad \ldots\ldots\ldots\ldots \quad (55)$$

and easily find on taking the real part of (54), and on making use of formulae (3), that at very great distances compared with the wave-length

$$E_\vartheta \sim H \sim \frac{P(\theta)}{r\sin\theta}\cos(\kappa r - wt) + \frac{\phi(\theta)}{r\sin\theta}\sin(\kappa r - wt), \quad \ldots\ldots\ldots \quad (56)$$

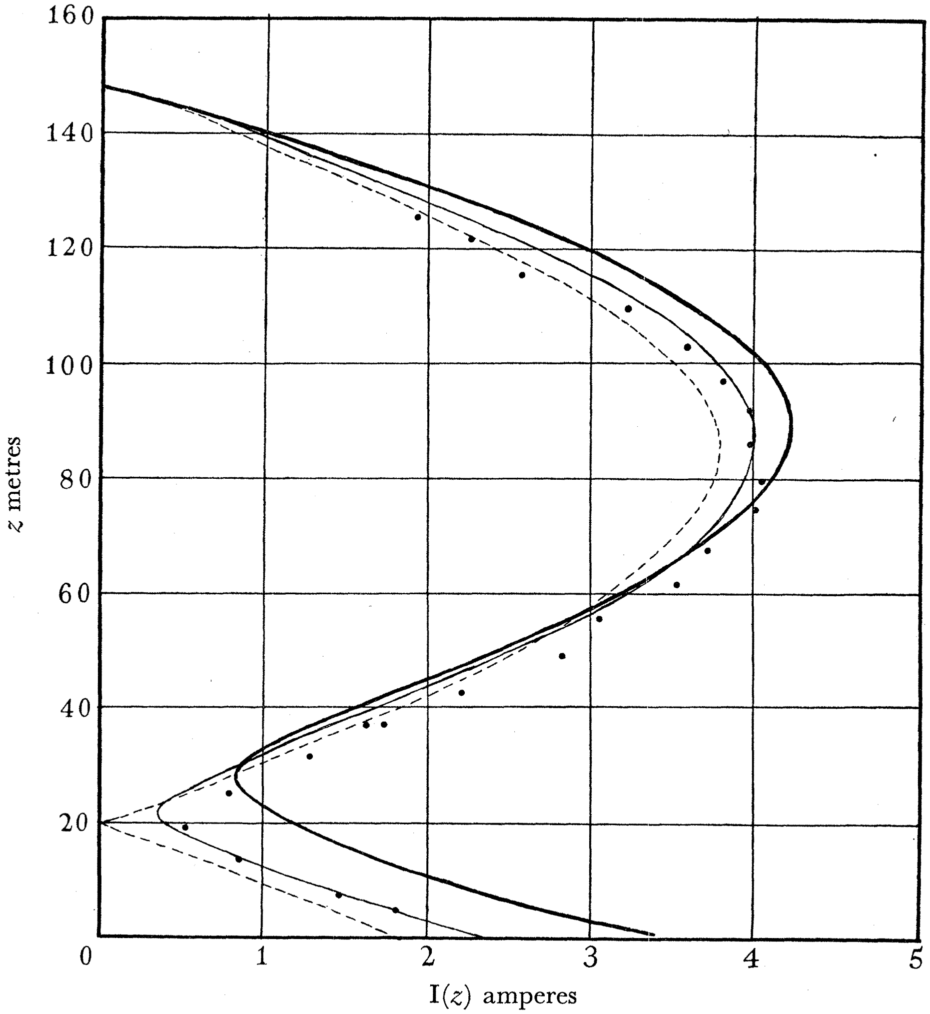

FIG. 5—*Root-mean-square Current Distribution Along a Perfectly Conducting Antenna*—From the entries of Table I the root-mean-square of current,

$$|w\ (\zeta)|/\sqrt{2} = (A/\sqrt{2})\ \{w''^2\ (\zeta) + w'^2\ (\zeta)\}^{\frac{1}{2}}, \qquad (\zeta = 2\pi\ (l - z)/\lambda),$$

is calculated at $\alpha = 2\pi l/\lambda = 3\cdot64$, for $\log\ (2l/a) = 4\cdot5$ (thick curve), $\log\ (2l/a) = 10\cdot58$ (thin curve), and $\log\ (2l/a) \to \infty$ (dotted curve). The constant A in each case is adjusted to a radiation output of 1 kW as explained under fig. 6. The black dots represent Professor PEDERSEN's observations on the Copenhagen antenna taken from fig. 30 of his paper (PEDERSEN, 1935). As the radiated power is not quoted, the scale of current has been reduced proportionally to fit the theoretical curve for $\log\ (2l/a) = 10\cdot58$ at 1 kW output at the base of the antenna. The remaining points fall fairly well on the corresponding theoretical (thin) curve. It will be noticed that as $\log\ (2l/a)$ decreases, the current minima become less sharply marked and tend to move towards the upper end of the antenna, where the deviation from the sinusoidal distribution is much less marked than at the base of the antenna.

TABLE II

$S' = w'_1 + w'_2 + w'_3$ $S'' = w_1'' + w''_2 + w''_3$ $\tan s = S''/S'$

ζ	S''	w'_1	w'_2	w'_3	S'	$w'(\zeta)$	$w''(\zeta)$	$\|w(\zeta)\|$	$\frac{3}{4}\sin\zeta$
0·0	0·000	0·000	0·000	0·000	0·000	0·0000	0·0000	0·0000	0·000
0·3	0·038	1·681	0·005	−0·018	1·668	−0·00578	0·2487	0·2489	0·222
0·6	0·163	3·837	0·018	−0·084	3·771	−0·01092	0·4587	0·4588	0·424
0·9	0·298	5·666	0·041	−0·192	5·515	−0·01432	0·6284	0·6286	0·588
1·2	0·638	7·052	0·076	−0·316	6·812	−0·01003	0·7396	0·7397	0·700
1·5	0·963	7·833	0·117	−0·550	7·400	−0·00345	0·7874	0·7874	0·747
1·8	1·313	7·920	0·161	−0·650	7·431	+0·00612	0·7615	0·7615	0·730
2·1	1·672	7·298	0·202	−0·801	6·699	0·01914	0·6699	0·6703	0·648
2·4	1·994	5·988	0·235	−0·819	5·404	0·03348	0·5167	0·5178	0·506
2·5	2·102	5·420	0·241	−0·805	4·856	0·03867	0·4512	0·4528	0·449
2·6	2·186	4·796	0·246	−0·789	4·253	0·04349	0·3880	0·3904	0·386
2·7	2·277	4·122	0·247	−0·745	3·624	0·04863	0·3170	0·3208	0·320
2·8	2·341	3·400	0·252	−0·696	2·956	0·05315	0·2630	0·2683	0·266
2·9	2·411	2·644	0·243	−0·668	2·219	0·05815	0·1662	0·1761	0·180
3·0	2·472	1·853	0·235	−0·513	1·575	0·06244	0·0872	0·1072	0·106
3·1	2·520	1·051	0·226	−0·381	0·896	0·06659	0·0063	0·0669	0·0312
3·3	2·582	−0·599	0·193	−0·023	−0·429	0·07372	−0·1562	0·1732	−0·118
3·5	2·603	−2·237	0·144	+0·549	−1·544	0·07883	−0·3183	0·3276	−0·262
3·6	2·594	−3·031	0·120	1·247	−1·664	0·07911	−0·4068	0·4143	−0·332

Current distribtion along antenna for $\log(2l/a) = 10\cdot58, \alpha = 3\cdot64$, $A = 1$, $|M| = 17\cdot75$, $m° = 8° - 22'$. It is evident from formula (50) that the constituents w''_1, w''_2, w''_3 are independent of the antenna dimensions. They are summed from Table I and entered in the second column. The last column gives the current distribution $w(\zeta)$ corresponding to $\log(2l/a) \rightarrow \infty$ for the same amplitude constant $A = 1$. In this case it is easily seen that there is only one component of current which is sinusoidal.

where

$$\left.\begin{aligned} P(\theta) &= 2A\{\cos(\alpha\cos\theta) - \cos\alpha\} + 2\int_0 \{\cos(\kappa\lambda\cos\theta) - \cos\kappa\lambda\}\,\phi'(\lambda)\,d\lambda \\ \phi(\theta) &= -2\int_0^l \{\cos(\kappa\lambda\cos\theta) - \cos\kappa\lambda\}\,\phi''(\lambda)\,d\lambda \end{aligned}\right\}. \quad (57)$$

To evaluate the integrals in (57) we write

$$C(\lambda) - iS(\lambda) = \frac{e^{-i\kappa(l-\lambda)}}{\{(l-\lambda)^2 + a^2\}^{\frac{1}{2}}} + \frac{e^{-i\kappa(l+\lambda)}}{\{(l+\lambda)^2 + a^2\}^{\frac{1}{2}}} - 2\cos\alpha\,\frac{e^{-i\kappa\lambda}}{\{\lambda^2 + a^2\}^{\frac{1}{2}}}, \quad (58)$$

so that, from (46) and (58),

$$\left.\begin{aligned} \phi'(\lambda)\cos m - \phi''(\lambda)\sin m &= -\{\tfrac{1}{2}A/|M|\}\,C(\lambda) \\ \phi'(\lambda)\sin m + \phi''(\lambda)\cos m &= -\{\tfrac{1}{2}A/|M|\}\,S(\lambda) \end{aligned}\right\}. \quad . \quad . \quad (59)$$

By straightforward integrations with the aid of the formulae of § 4 we find

$$\left.\begin{aligned} \int_0^l \{\cos(\kappa\lambda\cos\theta) - \cos\kappa\lambda\}\,C(\lambda)\,d\lambda &= \{\cos(\alpha\cos\theta) - \cos\alpha\}\log\frac{4l}{a} + C(\theta) \\ \int_0^l \{\cos(\kappa\lambda\cos\theta) - \cos\kappa\lambda\}\,S(\lambda)\,d\lambda &= S(\theta) \end{aligned}\right\}, \quad (60)$$

where, in terms of the variable ζ defined by

$$\zeta = \alpha \sin^2 \tfrac{1}{2}\theta, \qquad \alpha - \zeta = \alpha \cos^2 \tfrac{1}{2}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$\left.\begin{aligned} \mathrm{C}(\theta) = & -\tfrac{1}{2}\cos(\alpha - 2\zeta)\{\mathrm{S}_1 4(\alpha-\zeta) + \mathrm{S}_1(4\zeta)\} \\ & + \tfrac{1}{2}\sin(\alpha-2\zeta)\{\mathrm{S}_2 4(\alpha-\zeta) - \mathrm{S}_2(4\zeta)\} \\ & + \cos\alpha\{\mathrm{S}_1 2(\alpha-\zeta) + \mathrm{S}_1(2\zeta) + \mathrm{S}_1(2\alpha)\} + \mathrm{U}(\alpha) \\ \mathrm{S}(\theta) = & \tfrac{1}{2}\cos(\alpha-2\zeta)\{\mathrm{S}_2 4(\alpha-\zeta) + \mathrm{S}_2(4\zeta)\} \\ & + \tfrac{1}{2}\sin(\alpha-2\zeta)\{\mathrm{S}_1 4(\alpha-\zeta) - \mathrm{S}_1(4\zeta)\} \\ & - \cos\alpha\{\mathrm{S}_2 2(\alpha-\zeta) + \mathrm{S}_2(2\zeta) + \mathrm{S}_2(2\alpha)\} - \mathrm{V}(\alpha) \end{aligned}\right\} . \quad (62)$$

From (57) and (59) we find

$$\left.\begin{aligned} \frac{\mathrm{P}^2(\theta) + \phi^2(\theta)}{4\mathrm{A}^2} = & \{\cos(\alpha\cos\theta) - \cos\alpha\}^2 \left[1 - \frac{\cos m}{|\mathrm{M}|}\log\frac{4l}{a} + \left\{\frac{\log(4l/a)}{2|\mathrm{M}|}\right\}^2\right] \\ & - \frac{1}{|\mathrm{M}|}\left\{\cos m - \frac{1}{2|\mathrm{M}|}\log(4l/a)\right\}\{\cos(\alpha\cos\theta) - \cos\alpha\}\,\mathrm{C}(\theta) \\ & - \frac{1}{|\mathrm{M}|}\sin m\{\cos(\alpha\cos\theta) - \cos\alpha\}\,\mathrm{S}(\theta) \\ & + \frac{1}{4|\mathrm{M}|^2}\{\mathrm{C}^2(\theta) + \mathrm{S}^2(\theta)\} \end{aligned}\right\} . \quad (63)$$

To illustrate the nature of the radiation field from a perfectly conducting antenna, the functions C (θ) and S (θ), evaluated for the Copenhagen antenna are tabulated in Table II, and in figs. 5, 6, and 7 polar diagrams are drawn of the root-mean-square of the distant fields, *i.e.*, of

$$\mathrm{E}_{\theta\,\mathrm{r.m.s.}} \sim \mathrm{H}_{\mathrm{r.m.s.}} \sim (1/\sqrt{2})\{\mathrm{P}^2(\theta) + \phi^2(\theta)\}/r\sin\theta. \quad . \quad . \quad . \quad . \quad (64)$$

It will be noticed that a cone of zero field corresponding to the real root of $\cos(\alpha\cos\theta) = \cos\alpha$, according to the theory of the sine-wave antenna, is replaced by a minimum, agreeing in this respect with Pedersen's conclusions based on a modified line theory (PEDERSEN, 1935, p. 42, and figs. 35*a* and 35*b*). To illustrate the effect of antenna dimensions on the radiation field, a polar diagram is drawn for the same value of $\alpha = 3{\cdot}64$, the same power output of 1 kW, but for $\log(2l/a) = 4{\cdot}5$. It will be noticed that the loop in the interval $0 < \theta < 40°$ is considerably accentuated, as shown in greater detail in fig. 7, while the electric vector at $\theta = 90°$ is somewhat greater.

Recently aeroplane observations have been made of the radiation field over tall antenna towers, and the absence of the cone of zero field predicted by the sine-wave theory has been noted (BALLANTINE, 1934, fig. 41, p. 624). Such observations taken at a constant height above the earth are best studied in the light of formula (11), as the polar formula (64) cannot be considered sufficiently accurate unless r is a very large multiple of l.

7—Calculation of Radiation Resistance

We shall suppose (fig. 2) that the antenna is supplied by current by a " feeder " at height h from the earth and that the input current $w(h)$ is measured at this point.

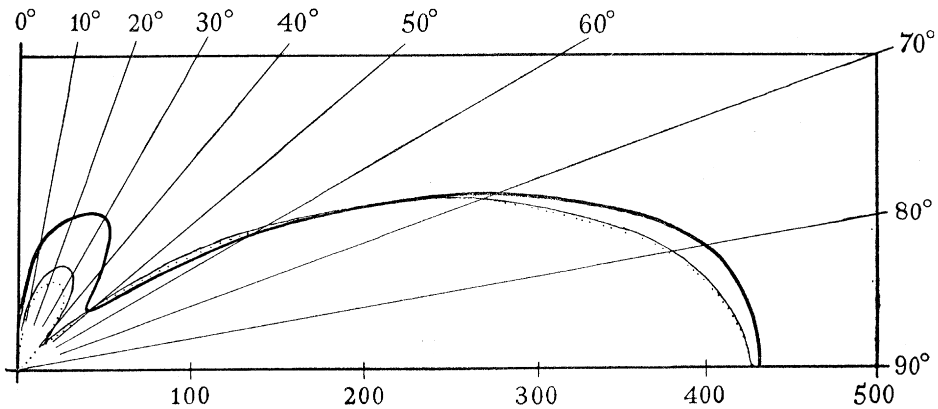

Fig. 6—*Polar Diagram of Radiation Field*—From the data of Table III the root-mean-squares of the electric vectors, $E_{\theta\,\mathrm{r.m.s.}}$ in millivolts per metre, given by equations (63) and (64), are plotted against the angle θ from the vertical for $\log(2l/a) = 4\cdot5$ (thick line), $\log(2l/a) = 10\cdot58$ (thin line), and $\log(2l/a) \rightarrow \infty$ (dotted curve), at a distance of 1 km. from the base of the antenna. In each case the constant A has been adjusted to correspond to a radiation output $[dW/dr] = 1\ \mathrm{kW} = 10^{10}$ ergs/sec. According to equations (67) and (76), we have

$$[dW/dr] = \tfrac{1}{2}A^2R(h)\{w'^2(h) + w''^2(h)\} = \tfrac{1}{2}A^2c\left[R(\alpha)\{R_1^2(\alpha) + R_2^2(\alpha)\} + \int_0^\alpha f(\zeta)\,d\zeta/\zeta\right], \quad \text{(i)}$$

while (73), (74), and (75) give on writing $\theta = \tfrac{1}{2}\pi, f(\tfrac{1}{2}\alpha) = 0$, and

$$\{P^2(\tfrac{1}{2}\pi) + \phi^2(\tfrac{1}{2}\pi)\}/(4A^2) = (1-\cos\alpha)^2\{R^2_1(\alpha) + R_2^2(\alpha)\}, \quad \text{(ii)}$$

so that $R_1^2(\alpha) + R_2^2(\alpha)$ is known from the last line of Table III. In this way we easily find the following values for A and $E_{\frac{1}{2}\pi\,\mathrm{r.m.s.}}$ at 1 km. according to (64). For $\alpha = 3\cdot64$, Table D gives by interpolation $R(\alpha) = 2\cdot353$. We thus find on passing from C.G.S. to practical electrical units the following values for the current-amplitude constant A and for the r.m.s. value of the electric field at $\theta = 90°$ the following values :

$\log(2l/a)$	$R_1^2(\alpha) + R_2^2(\alpha)$	$\int_0^\alpha f(\zeta)\,d\zeta/\zeta$	A (amps)	$E_{\frac{1}{2}\pi\,\mathrm{r.m.s.}}$ (mV/metre)
4·5	0·526	—0·072	7·55	433
10·58	0·556	—0·017	7·18	425
	0·562	0·000	7·10	424

We shall disregard the stray field due to the " feeder " since in practice current is supplied from a concentric main the outer conductor of which is earthed. When h is small compared with the wave-length, the configuration of the electric field at the base of the antenna may, with sufficient accuracy, be described in terms of a potential, so that it is legitimate to refer to a measurable voltage $V(h)$ between the point h

and the earth in its immediate neighbourhood. In these circumstances the rate of power input into the antenna at any instant is given by

$$dW/dt = V(h) \cdot w(h). \quad (65)$$

Similarly, we are justified in describing the relation between current and "potential" at $z = h$ in terms of an input impedance

$$Z(h) = R(h) + iX(h), \quad (66)$$

defined by the relation

$$V(h) = Z(h) \cdot w(h). \quad (67)$$

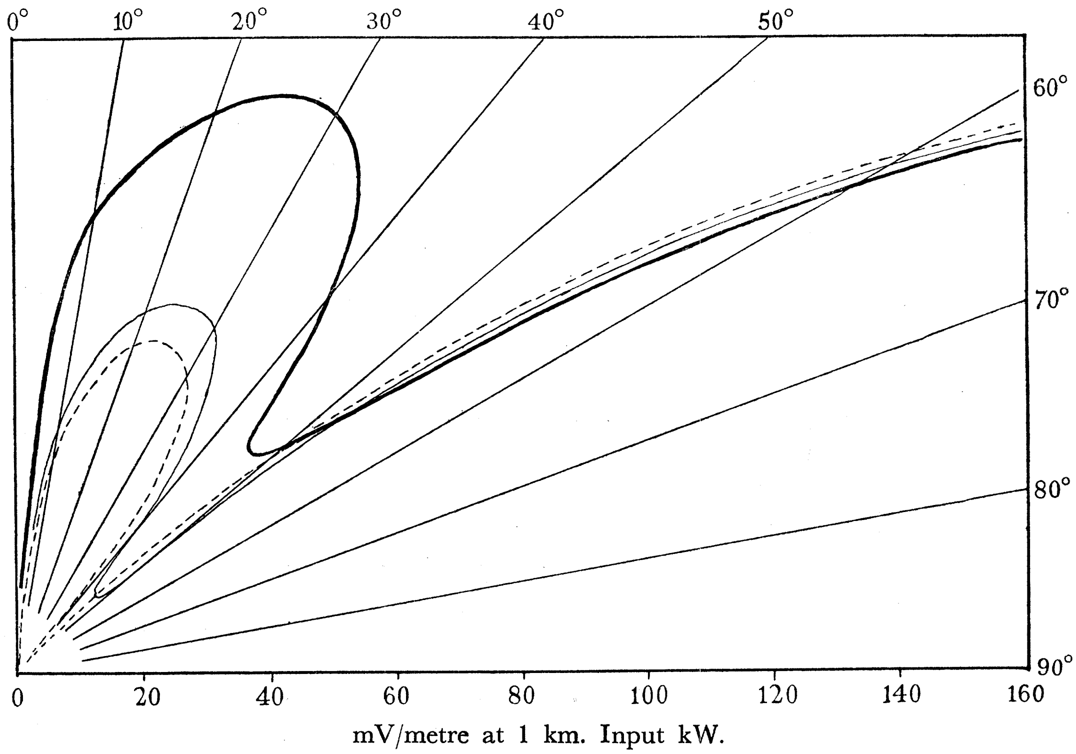

FIG. 7—*Polar Diagram of Radiation Field*—Enlarged part of the polar diagram of fig. 6 giving E_θ 1 r.m.s. in millivolts per metre at 1 km. distance for 1 kW radiation output at $\alpha = 3 \cdot 64$. The thick curve corresponds to $\log(2l/a) = 4 \cdot 5$, the thin curve to $\log(2l/a) = 10 \cdot 58$, and the dotted curve to $\log(2l/a) = \infty$.

According to equation (52), $w(h)$ has two components in phase quadrature so that, introducing the time-factor e^{iwt},

$$w(h) = A\, w'(h) - iw''(h)\, e^{iwt}. \quad (68)$$

On introducing the real parts of V (h) and $w(h)$ into (65), and taking a time-average, X (h) disappears and we find

$$[dW/dt]_{av.} = \tfrac{1}{2}A^2R(h)\,\{w'^2(h) + w''^2(h)\}. \quad (69)$$

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TABLE III—POLAR FIELD DISTRIBUTION FOR $\alpha = 3\cdot64$

θ	$\zeta = \alpha \sin^2 \frac{1}{2}\theta$	$\text{Cos}\,(\alpha \cos \alpha) - \cos \alpha$	$C(\theta)$	$S(\theta)$	$\log(2l/a) = 4\cdot5$: $\frac{\{P^2(\theta)+\phi^2(\theta)\}^{\frac{1}{2}}}{2A \sin \theta}$	$\log(2l/a) = 4\cdot5$: $f(\theta)$	$\log(2l/a) = 10\cdot58$: $\frac{\{P^2(\theta)+\phi^2(\theta)\}^{\frac{1}{2}}}{2A \sin \theta}$	$\log(2l/a) = 10\cdot58$: $f(\theta)$	$\log(2l/a)\rightarrow\infty$: $\frac{\{P^2(\theta)+\phi^2(\theta)\}^{\frac{1}{2}}}{2A \sin \theta}$
5°	0·0069	—0·0066	0·040	0·016	0·079	0·000024	0·072	0·000015	0·057
10°	0·0277	—0·026	0·12	0·080	0·211	0·00052	0·131	0·00070	0·112
20°	0·110	—0·083	0·35	0·33	0·273	0·0052	0·199	0·00084	0·182
30°	0·244	—0·122	0·61	0·59	0·297	0·0143	0·209	0·00027	0·183
35°	0·329	—0·109	0·67	0·86	0·276	0·0189	0·175	0·0035	0·142
37°·5	0·376	—0·091	0·66	1·00	0·275	0·0202	0·147	0·0034	0·112
40°	0·426	—0·061	0·62	1·16	0·228	0·0195	0·111	0·0030	0·0710
42°·5	0·478	—0·018	0·54	1·33	0·194	0·0170	0·073	0·0022	0·0200
45°	0·532	+0·036	0·43	1·53	0·160	0·0122	0·055	0·00072	0·0372
50°	0·650	0·181	0·09	1·95	0·158	—0·0030	0·162	—0·0028	0·179
60°	0·910	0·630	—1·09	2·86	0·452	—0·066	0·526	—0·0120	0·546
70°	1·198	1·198	—2·73	3·86	0·853	—0·095	0·938	—0·0200	0·956
80°	1·503	1·684	—4·15	4·56	1·191	—0·086	1·267	—0·0160	1·282
90°	1·820	1·878	—4·73	4·81	1·360	0·000	1·400	0·000	1·408

The functions $C(\theta)$ and $S(\theta)$ are defined by equations (63), while $f(\theta)$, defined by equation (77) below, when integrated by quadratures as indicated in (78), makes it possible to obtain an exact value for the radiation resistance.

The rate of radiation of energy from the antenna through a distant hemisphere, is, according to Poynting's theorem,

$$dW/dt = \tfrac{1}{2}c\int_0^{\frac{1}{2}\pi} E_\theta \,.\, Hr^2 \sin\theta \, d\theta. \qquad (70)$$

On introducing the values of E_θ and H for the distant fields from (56) and taking a time-average, we find

$$[dW/dt]_{\text{av.}} = \tfrac{1}{4}c\int_0^{\frac{1}{2}\pi} \{P^2(\theta) + \phi^2(\theta)\}\frac{d\theta}{\sin\theta}. \qquad (71)$$

In the perfectly conducting antenna we may equate (69) and (71)* with the result that on introducing $P^2(\theta) + \phi^2(\theta)$ from (63) and making use of (34) we find the following expression for the radiation resistance

$$\frac{R(h)}{c} = \frac{1}{\{w'^2(h) + w''^2(h)\}}$$
$$\times\left[\begin{aligned}&\left\{1 - \frac{\cos m}{|M|}\log(4l/a) + \left\{\frac{\log(4l/a)}{2|M|}\right\}^2\right\}R(\alpha)\\ &-\frac{2}{|M|}\left\{\cos m - \frac{1}{2|M|}\log(4l/a)\right\}\int_0^{\frac{1}{2}\pi}\{\cos(\alpha\cos\theta) - \cos\alpha\}\frac{C(\theta)\,d\theta}{\sin\theta}\\ &-\frac{2}{|M|}\sin m\int_0^{\frac{1}{2}\pi}\{\cos(\alpha\cos\theta) - \cos\alpha\}\frac{S(\theta)\,d\theta}{\sin\theta}\\ &\qquad\qquad + \frac{1}{2|M|^2}\int_0^{\frac{1}{2}\pi}\{C^2(\theta) + S^2(\theta)\}\frac{d\theta}{\sin\theta}\end{aligned}\right]. \qquad (72)$$

The integrals in (72) are best dealt with in terms of the variable $\zeta = \alpha\sin^2\frac{1}{2}\theta$ in terms of which $C(\theta)$ and $S(\theta)$ are expressed in (62). We notice that $C(\zeta) = C(\alpha - \zeta)$, $S(\zeta) = S(\alpha - \zeta)$, while

$$\cos(\alpha\cos\theta) - \cos\alpha = \sin\alpha\sin 2\zeta - \cos\alpha(1 - \cos 2\zeta)$$

has the same property. The above integrals are of the type

$$\int_0^{\frac{1}{2}\pi} f(\theta)\frac{d\theta}{\sin\theta} = \tfrac{1}{2}\alpha\int_0^{\frac{1}{2}\alpha}\frac{f(\zeta)\,d\zeta}{\zeta(\alpha - \zeta)} = \tfrac{1}{2}\int_0^{\frac{1}{2}\alpha} f(\zeta)\left\{\frac{1}{\zeta} + \frac{1}{\alpha - \zeta}\right\}d\zeta.$$

Since $f(\zeta) = f(\alpha - \zeta)$, by an obvious transformation of the last integral, we find

$$\int_0^{\frac{1}{2}\pi} f(\theta)\frac{d\theta}{\sin\theta} = \tfrac{1}{2}\int_0^{\alpha} f(\zeta)\frac{d\zeta}{\zeta}. \qquad (73)$$

* In the case of the Copenhagen antenna rough calculations indicate that the ohmic resistance characteristic of copper or aluminium is less than 1% of the radiation resistance at $l/\lambda = 0{\cdot}581$.

By this means the first two integrals in (72) may be made to depend on integrals of the type

$$\int_0^\alpha \sin n\zeta S_1(\zeta)\frac{d\zeta}{\zeta}, \quad \int_0^\alpha \sin n\zeta S_2(\zeta)\frac{d\zeta}{\zeta}$$

and $\qquad n = 0, 1, 2, 3, \ldots \quad (74)$

$$\int_0^\alpha \cos n\zeta S_2(\zeta)\frac{d\zeta}{\zeta}, \quad \int_0^\alpha \cos n\zeta S_2(\zeta)\frac{d\zeta}{\zeta}$$

which, however, have not been tabulated. The last integral in (72), though of lesser importance, requires the tabulation of still more complicated functions of α.

Over the broadcasting range $0 < \alpha < 4\cdot0$ when the secondary loops illustrated in fig. 6 are entirely absent or relatively small, we notice that if the electric vectors E_θ are made to coincide at $\theta = 90°$, the polar diagram of the perfectly conducting antenna differs little from that of the sine-wave antenna for which

$$P'^2(\theta) + \phi'^2(\theta) = 4A'^2\{\cos(\alpha\cos\theta) - \cos\alpha\}^2, \quad \ldots \quad (75)$$

except in the neighbourhood of the minimum between the two loops should a subsidiary loop exist.* If we make

$$P'^2(\tfrac{1}{2}\pi) + \phi'^2(\tfrac{1}{2}\pi) = P^2(\tfrac{1}{2}\pi) + \phi^2(\tfrac{1}{2}\pi)$$

we easily find from (63) that

$$A'^2 = A^2\{R_1^2(\alpha) + R_2^2(\alpha)\}$$

when

$$\left.\begin{aligned} R_1 &= \cos m - \frac{1}{2|M|}\left\{\log(4l/a) + \frac{C(\frac{1}{2}\pi)}{1-\cos\alpha}\right\} \\ R_2 &= \sin m - \frac{1}{2|M|}\frac{S(\frac{1}{2}\pi)}{1-\cos\alpha} \end{aligned}\right\} \quad \ldots \quad (76)$$

We may then write

$$\{P^2(\theta) + \phi^2(\theta)\}/(4A^2) = \{R^2{}_1(\alpha) + R_2^2(\alpha)\}\{\cos(\alpha\cos\theta) - \cos\alpha\}^2 + f(\theta) \quad (77)$$

where $f(\theta)$ may be tabulated from (62) and (63) in terms of the variable $\zeta = \alpha\sin^2\frac{1}{2}\theta$.

On making use of (69), (71), (73), and (76) we obtain the approximate formula valid over the range $0 < \alpha < 4\cdot0$

$$\frac{R(h)}{c} = \frac{1}{w'^2(h) + w''^2(h)}\left\{\{R_1^2(\alpha) + R_2^2(\alpha)\}\,.\,R(\alpha) + \int_0^\alpha \frac{f(\zeta)\,d\zeta}{\zeta}\right\}. \quad (78)$$

If it is desired to compute radiation resistances for values of $\alpha > 4\cdot0$, integration by quadratures is most accurately effected by calculating A' from the equation

$$P'^2(\theta_m) + \phi'^2(\theta_m) = P^2(\theta_m) + \phi^2(\theta_m),$$

where θ_m is the value of θ corresponding to the greatest maximum of $\{P'^2(\theta) + \phi'^2(\theta)\}/\sin\theta$.

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When a polar diagram for $P^2(\theta) + \phi^2(\theta)$ has been computed from (63) a graph for $f(\zeta)$ is easily plotted from (77) and the last integral in (78) evaluated by quadratures. In a numerical example considered under fig. 6, it is shown that for $\alpha = 3{\cdot}64$ the neglect of this term gives values of the radiation resistance in excess of the correct value by $5{\cdot}8\%$ when $\log(2l/a) = 4{\cdot}5$ and by $1{\cdot}3\%$ when $\log(2l/a) = 10{\cdot}58$.

For purposes of computing the radiation resistance from (76) and (78), we find on writing $\theta = \frac{1}{2}\pi$ or $\zeta = \frac{1}{2}\alpha$ in formulae (62) that

$$\left.\begin{aligned} C(\tfrac{1}{2}\pi) &= -S_1(2\alpha) + \cos\alpha\,\{2S_1(\alpha) + S_1(2\alpha)\} + U(\alpha) \\ S(\tfrac{1}{2}\pi) &= S_2(2\alpha) - \cos\alpha\,\{2S_2(\alpha) + S_2(2\alpha)\} - V(\alpha) \end{aligned}\right\} \quad . \quad . \quad . \quad (79)$$

When $h \gg a$ but h/l is small of the order $0{\cdot}01$, we write, according to (47), $\alpha - \zeta = \kappa h$, with the result that the formulae (49) and (50) are considerably simplified. Since $\kappa h \ll 1$ we find from (51)

$$\left.\begin{aligned} S' &= \sin\alpha\,\{\log(l/a) - S_1(2\alpha)\} + V(\alpha) + 2\cos\alpha\sin\kappa h\log(l/h) \\ S'' &= \sin\alpha\,S_2(2\alpha) + U(\alpha) \end{aligned}\right\} . \quad (80)$$

so that, according to (53), with $\tan s = S''/S'$

$$w'(h) = \tfrac{1}{2}|S|/|M|\sin(s-m) \qquad w''(h) = \sin\alpha - \tfrac{1}{2}|S|/|M|\cos(s-m). \quad (81)$$

By means of formula (78) and the associated formulae (79), (80), and (81), the radiation resistance of a perfectly conducting antenna has been calculated for twenty values of α in the range $0 < \alpha < 4{\cdot}0$ for value of $\log(2l/a) = 4{\cdot}5$ and $\log(2l/a) = 10{\cdot}58$, while $h/l = 0{\cdot}01$ is assumed in both cases.* The results are tabulated with comments in Tables E, F, G of the appendix, and the radiation resistances, expressed in ohms in each case, are plotted on semi-logarithmic paper (fig. 8) as a function of $\alpha = 2\pi l/\lambda$, together with the radiation reactances which we now proceed to calculate.

8—Calculation of Radiation Reactance

As we have already intimated, the electric field in the neighbourhood of the base of the antenna throughout a region of linear dimensions, small with the wave-length, approximates closely to an electrostatic field characterized by a potential V. Referring to fig. 2, let AB be a line of force at any instant from a point (a, h) on the antenna to the perfectly conducting earth at B. If $V(h)$ be the "potential" between A and B, we may write

$$V(h) = -\int_A^B E_s\,ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

* Professor Pedersen has kindly informed the writer that in the Copenhagen antenna $h = 0{\cdot}92$ metre above the perfect earth and that the base of the antenna is directly on the perfect earth assumed to be the net of copper wires dug some 25 cm. into the ground.

where s is measured along this line of force and E_s is the tangential component of the electric field along it. At A the component $E_\rho\,(a, h)$ is normal to the antenna, so that by Gauss's theorem

$$E_\rho\,(a, h) = 2\sigma\,(h)/a, \qquad (83)$$

where $\sigma\,(h)$ is the charge per unit length on the antenna at $z = h$.

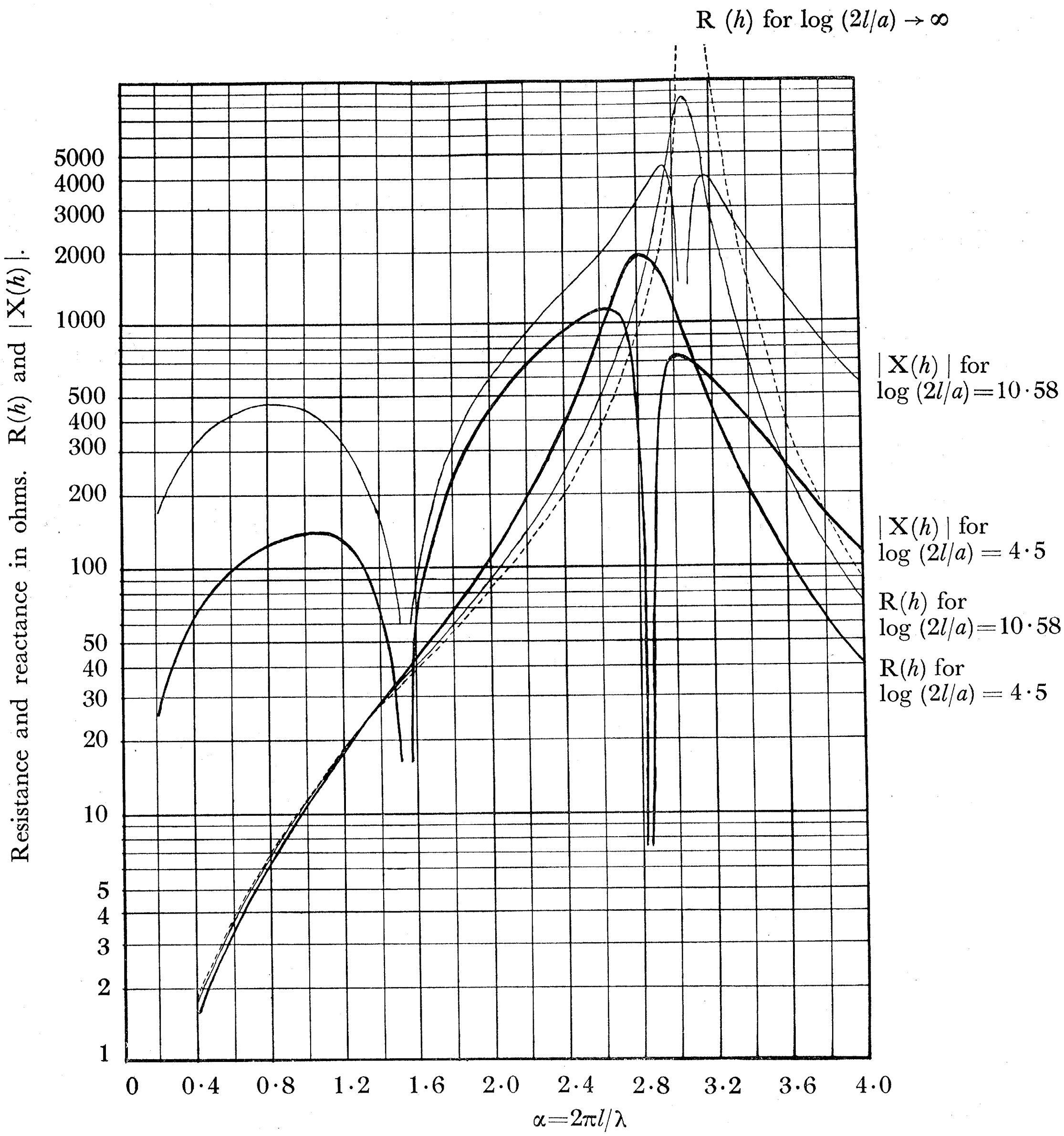

Fig. 8.—The impedance characteristics of a perfectly conducting, base insulated cylindrical antenna over a perfectly conducting earth, as worked out in Tables E, F, and G (*see* Appendix), are plotted on a semi-logarithmic scale for values of log $(2l/a) = 4\cdot 5$ (thick curves), $10\cdot 58$ (thin curves), and log $(2l/a) \to \infty$ (dotted curves) respectively.

We may thus write (82) in the form

$$V\,(h) = -\frac{2\sigma\,(h)}{a}\int_A^B \frac{E_s}{E_\rho\,(a, h)}\,ds. \qquad (84)$$

When the path of integration is very small compared with the wave-length, phase differences of E_s at different points of AB are negligible, so that $E_s/E_\rho\ (a, h)$ remains practically independent of the time. Similarly the position of the line of force AB remains very nearly unaltered throughout time, as in the corresponding electrostatic problem, when propagation effects are neglected. It follows that the integral (84) is very nearly independent of the time, and is of the nature of a real constant. We may thus write

$$V(h) = -B\sigma(h). \qquad (85)$$

where B is a real constant, and $h \ll$ wave-length.

At any point in space we have, according to equations (4),

$$\dot{E}_\rho(\rho, z) = -(c/\rho)\,\partial\psi/\partial z,$$

and at any point of the antenna, Gauss's theorem gives for the charge per unit length

$$E_\rho(a, z) = 2\sigma(z)/a,$$

from which it follows that

$$\dot{\sigma}(z) = -\tfrac{1}{2}c\,(\partial\psi/\partial z)_{\rho=a}. \qquad (86)$$

It is a simple matter to show that with solutions of the type (9) we are justified in writing $(\partial\psi/\partial z)_{\rho=a} = \partial(\psi)_{\rho=a}/\partial z$, and since for a thin antenna we have, according to equation (3), $w(z) = \frac{1}{2}(\psi)_{\rho=a}$, there results from (86)

$$\dot{\sigma}(z) + c\,dw/dz = 0, \qquad (87)$$

which also expresses continuity of charge over a section dz of the conductor.

On differentiating the expression (10) for $w(z)$, we easily find at $z = h$,

$$-\sigma(h) = A\cos\kappa(l-h) + \int_h^l \phi(\lambda)\cos\kappa(\lambda-h)\,d\lambda. \qquad (88)$$

On referring to (46) and (58), we have

$$\phi(\lambda) = -\tfrac{1}{2}A/|M|\,e^{im}\{C(\lambda) - iS(\lambda)\}.$$

We then find by the aid of the formulae of § 4 that when h/l is of the order of 0·01, a sufficiently accurate expression for the integral in (88) may be expressed in terms of C′ and C″ where

$$\int_h^l \cos\kappa(\lambda-h)\{C(\lambda) - iS(\lambda)\}\,d\lambda = -(C' + iC''),$$

and

$$\left.\begin{aligned} C' &= -\cos\alpha\{\log(l/a) - S(2\alpha)\} + U(\alpha) + 2\cos\alpha\log(\tfrac{1}{2}l/h) \\ C'' &= \cos\alpha S_2(2\alpha) + V(\alpha) \end{aligned}\right\}. \qquad (89)$$

If we now write $\tan c = C''/C'$, we easily find

$$\int_h^l \phi(\lambda)\cos\kappa(\lambda-h)\,d\lambda = \frac{A\,|C|}{2\,|M|}\,e^{i(c+m)}.$$

It is thus seen that $\sigma(h)$ is complex, so that if we denote

$$-\sigma(h) = A\{v'(h) - iv''(h)\}, \quad \ldots \ldots \ldots \quad (90)$$

we easily find

$$v'(h) = \cos\alpha + \tfrac{1}{2}|C|/|M| \cos(c+m), \quad v''(h) = -\tfrac{1}{2}|C|/|M| \sin(c+m). \quad (91)$$

We are now in a position to calculate the antenna reactance. From (67) and (85) we have

$$Z(h) \,.\, w(h) = -B\sigma(h),$$

which gives, on introducing the complex components of $\sigma(h)$ and $w(h)$ from (90) and (52),

$$Z(h) = B\{v'(h) - iv''(h)\}/\{w'(h) - iw''(h)\}. \quad \ldots \ldots \quad (92)$$

If we now denote

$$w''(h)/w'(h) = \tan w \quad \text{and} \quad v''(h)/v'(h) = \tan v, \quad \ldots \ldots \quad (93)$$

we have from (92)

$$Z(h) = R(h) + iX(h) = B\,|v(h)|/|w(h)| \,.\, e^{i(w-v)},$$

from which we immediately deduce, since B is a real constant, that

$$X(h) = R(h) \tan(w - v). \quad \ldots \ldots \ldots \ldots \quad (94)$$

Computations of the radiation impedances of the perfectly conducting antennae for which the radiation resistances were calculated and plotted from the formulae of § 7 are tabulated in Tables E, F, G of the appendix, and the reactances, $|X(h)|$, are plotted on the same semi-logarithmic chart (fig. 8.) There exists, unfortunately, very little experimental data on the radiation resistances and reactions of antennae of uniform cross-section over a large range of values of $\alpha = 2\pi l/\lambda$. It may be noted, however, that the results plotted in fig. 8 for $\log(2l/a) = 4\cdot5$ are in good general agreement with the characteristics of doubly-tapered antennae described in a recent paper by CHAMBERLAIN and LODGE,* reproduced in fig. (i). (*See* Appendix.)

Note on Very Thin Antennae

It is of some interest to note the limiting forms which the formulae of this paper take as $\log(2l/a) \to \infty$. In these circumstances

$$m \to 0 \quad \text{and} \quad |M| \sim 2\{\log(2l/a) - 1\},$$

for all finite values of α.

* CHAMBERLAIN and LODGE (1936). Fig. 5 of the paper referred to is here reproduced as fig. (i) in the Appendix, with the kind permission of the authors, the American Institute of Radio Engineers, and the Radio Club of America, in whose Proceedings (Vol. 11, November, 1934) the diagram of antennae characteristics was first published.

From (51) and (53) we find that $w'(\zeta) \to 0$ and $w''(\zeta) \to \frac{3}{4} \sin \zeta$, so that

$$w(\zeta) \to -\tfrac{3}{4} i \mathrm{A} \sin \kappa (l - z). \qquad (95)$$

Formula (78) for the radiation resistance is then easily found to reduce to

$$\mathrm{R}(h)/c \to \mathrm{R}(\alpha)/\sin^2 \alpha, \qquad (96)$$

as $\log(2l/a) \to \infty$.

From (89) we notice that $c \to 0$, so that according to (91), $v''(h) \to 0$ and $v'(h) \to \frac{3}{4} \cos \alpha$. Thus, in general, if $\cos \alpha \neq 0$, $\tan v \to \pm \varepsilon$ as $\varepsilon \to 0$ and thus $v \to 0$ or π. Since $\tan w \to \infty$, $w \to \frac{1}{2}\pi$, so that according to (94)

$$\mathrm{X}(h)/\mathrm{R}(h) \to \pm \infty, \qquad (97)$$

as $\log(2l/a) \to \infty$ for all values of α. Apparent exceptions when $\cos \alpha = 0$ require a closer scrutiny of the limiting forms. If we write $\cos \alpha = 0$ before making $\log(2l/a) \to \infty$ in equations (91) we find that $\tan c = -\mathrm{S}_1(4\alpha)/\mathrm{S}_2(4\alpha)$. Then if we make $\log(2l/a) \to \infty$ we deduce that $\mathrm{X}(h)/\mathrm{R}(h) \to \mathrm{S}_2(4\alpha)/\mathrm{S}_1(4\alpha)$, where α is a root of $\cos \alpha = 0$. But the slightest variation of α on either side of the root sends the value of X/R to $\neq \infty$, so that the reactances for these exceptional values of α are indeterminate. It thus appears from the above formulae that the perfectly conducting antenna approaches more and more closely to the sine-wave antenna as $\log(2l/a) \to \infty$, but that the impedance is infinite for all values of $\alpha = 2\pi l/\lambda$ except for values corresponding to $\cos \alpha = 0$ when the reactance is indeterminate. Such an antenna is not practically realizable.

We may note here that the procedure of this paper is equally applicable to the theory of the perfectly conducting receiving antenna. If $\mathrm{E}_0(z)$ is the vertical component of the incident wave, this term must be inserted on the right-hand side of equation (11) and $\mathrm{E}(z)$ made to vanish at $\rho = a$ for all values of z, while A is determined by the impedance of the receiving apparatus at the antenna base. It is at once evident that the solution for $\phi(\lambda)$ will not be the same as for the transmitting antenna, so that the radiation impedance of a receiving antenna will not be the same as that of a transmitting antenna of the same dimensions.

Needless to remark, the results of this paper and the antenna characteristics given in fig. 8 will be considerably modified by taking into account the effect of an imperfectly conducting dielectric earth and to a much lesser extent by the ohmic resistance of the conductor. The writer has considered it of some importance to have available as a standard of reference the radiation characteristics of the perfectly conducting antenna over a perfectly conducting earth, as several features of its behaviour formerly ascribed to earth currents turn out to be inherent in the solution of the electromagnetic equations with due regard to the boundary conditions along the conductor.

APPENDIX

Table A

$$S_1(x) = lgn\,(x) + C - Ci\,(x) = lgn\,(x) + 0\cdot5772157 - Ci\,(x)^*$$

x	S₁ (x)	x	S₁ (x)	x	S₁ (x)	x	S₁ (x)	x	S₁ (x)	x	S₁ (x)
0·0	0·00000	5·0	2·37669	10·0	2·92527	15·0	3·23899	20·0	3·52853	25·0	3·80295
0·1	0·00249	5·1	2·38994	10·1	2·94327	15·1	3·25090	20·1	3·53173		
0·2	0·00998	5·2	2·40113	10·2	2·96050	15·2	3·26308	20·2	3·53535		
0·3	0·02241	5·3	2·41044	10·3	2·97688	15·3	3·27552	20·3	3·53946		
0·4	0·03973	5·4	2·41801	10·4	2·99234	15·4	3·28814	20·4	3·54402		
0·5	0·06185	5·5	2·42402	10·5	3·00688	15·5	3·30087	20·5	3·54905		
0·6	0·08866	5·6	2·42866	10·6	3·02045	15·6	3·31363	20·6	3·55456		
0·7	0·12002	5·7	2·43210	10·7	3·03300	15·7	3·32641	20·7	3·56049		
0·8	0·15579	5·8	2·43452	10·8	3·04457	15·8	3·33911	20·8	3·56687		
0·9	0·19578	5·9	2·43610	10·9	3·05514	15·9	3·35167	20·9	3·57368		
1·0	0·23981	6·0	2·43704	11·0	3·06467	16·0	3·36401	21·0	3·58085		
1·1	0·28766	6·1	2·43749	11·1	3·07323	16·1	3·37612	21·1	3·58840		
1·2	0·33908	6·2	2·43764	11·2	3·08083	16·2	3·38790	21·2	3·59629		
1·3	0·39384	6·3	2·43766	11·3	3·08749	16·3	3·39932	21·3	3·60446		
1·4	0·45168	6·4	2·43770	11·4	3·09322	16·4	3·41032	21·4	3·61288		
1·5	0·51233	6·5	2·43792	11·5	3·09814	16·5	3·42088	21·5	3·62155		
1·6	0·57549	6·6	2·43847	11·6	3·10225	16·6	3·43096	21·6	3·63037		
1·7	0·64088	6·7	2·43947	11·7	3·10561	16·7	3·44050	21·7	3·63935		
1·8	0·70820	6·8	2·44106	11·8	3·10828	16·8	3·44947	21·8	3·64842		
1·9	0·77713	6·9	2·44335	11·9	3·11038	16·9	3·45788	21·9	3·65751		
2·0	0·84739	7·0	2·44643	12·0	3·11190	17·0	3·46568	22·0	3·66662		
2·1	0·91865	7·1	2·45040	12·1	3·11301	17·1	3·47288	22·1	3·67568		
2·2	0·99060	7·2	2·45534	12·2	3·11370	17·2	3·47945	22·2	3·68465		
2·3	1·06295	7·3	2·46130	12·3	3·11412	17·3	3·48543	22·3	3·69348		
2·4	1·13540	7·4	2·46834	12·4	3·11429	17·4	3·49077	22·4	3·70216		
2·5	1·20764	7·5	2·47649	12·5	3·11436	17·5	3·49553	22·5	3·71059		
2·6	1·27939	7·6	2·48577	12·6	3·11437	17·6	3·49969	22·6	3·71879		
2·7	1·35038	7·7	2·49619	12·7	3·11438	17·7	3·50330	22·7	3·72670		
2·8	1·42035	7·8	2·50775	12·8	3·11453	17·8	3·50639	22·8	3·73427		
2·9	1·48903	7·9	2·52044	12·9	3·11484	17·9	3·50895	22·9	3·74153		

3·0	1·55620	8·0	2·53423	13·0	3·11540	18·0	3·51107	23·0	3·74838		
3·1	1·62163	8·1	2·54906	13·1	3·11628	18·1	3·51276	23·1	3·75483		
3·2	1·68511	8·2	2·56491	13·2	3·11754	18·2	3·51404	23·2	3·76089		
3·3	1·74646	8·3	2·58171	13·3	3·11924	18·3	3·51500	23·3	3·76651		
3·4	1·80552	8·4	2·59938	13·4	3·12142	18·4	3·51568	23·4	3·77170		
3·5	1·86211	8·5	2·61786	13·5	3·12414	18·5	3·51610	23·5	3·77644		
3·6	1·91613	8·6	2·63704	13·6	3·12745	18·6	3·51633	23·6	3·78072		
3·7	1·96745	8·7	2·65686	13·7	3·13134	18·7	3·51645	23·7	3·78459		
3·8	2·01600	8·8	2·67721	13·8	3·13587	18·8	3·51648	23·8	3·78801		
3·9	2·06170	8·9	2·69799	13·9	3·14104	18·9	3·51648	23·9	3·79101		
4·0	2·10449	9·0	2·71909	14·0	3·14688	19·0	3·51650	24·0	3·79360		
4·1	2·14438	9·1	2·74042	14·1	3·15338	19·1	3·51661	24·1	3·79582		
4·2	2·18131	9·2	2·76186	14·2	3·16054	19·2	3·51685	24·2	3·79767		
4·3	2·21535	9·3	2·78332	14·3	3·16835	19·3	3·51727	24·3	3·79917		
4·4	2·24648	9·4	2·80468	14·4	3·17677	19·4	3·51790	24·4	3·80036		
4·5	2·27479	9·5	2·82583	14·5	3·18583	19·5	3·51879	24·5	3·80129		
4·6	2·30033	9·6	2·84669	14·6	3·19545	19·6	3·52002	24·6	3·80197		
4·7	2·32317	9·7	2·86715	14·7	3·20564	19·7	3·52156	24·7	3·80243		
4·8	2·34344	9·8	2·88712	14·8	3·21630	19·8	3·52348	24·8	3·80271		
4·9	2·36124	9·9	2·90651	14·9	3·22746	19·9	3·52578	24·9	3·80288	50·0	4·49486

* In calculating Tables A and B we have applied KEIKITIRO TANI: Tables of *si* (x) and *ci* (x) for the range $x = 0$ to $x = 50$ (Tokyo, 1931). [$ci\ (x) = \mathrm{C}i\ (x)$].

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TABLE B

$$S_2(x) = \mathrm{Si}(x) \left[= \frac{\pi}{2} + si(x) \right]$$

x	Si (x)	x	Si (x)	x	Si (x)	x	Si (x)	x	Si (x)	x	Si (x)
0·0	0·00000	5·0	1·54993	10·0	1·65835	15·0	1·61819	20·0	1·54824	25·0	1·53148
0·1	0·09994	5·1	1·53125	10·1	1·65253	15·1	1·62226	20·1	1·55289		
0·2	0·19956	5·2	1·51367	10·2	1·64600	15·2	1·62575	20·2	1·55767		
0·3	0·29850	5·3	1·49732	10·3	1·63883	15·3	1·62865	20·3	1·56253		
0·4	0·39646	5·4	1·48230	10·4	1·63112	15·4	1·63093	20·4	1·56743		
0·5	0·49311	5·5	1·46872	10·5	1·62294	15·5	1·63258	20·5	1·57232		
0·6	0·58813	5·6	1·45667	10·6	1·61439	15·6	1·63359	20·6	1·57714		
0·7	0·68122	5·7	1·44620	10·7	1·60556	15·7	1·63396	20·7	1·58186		
0·8	0·77210	5·8	1·43736	10·8	1·59654	15·8	1·63370	20·8	1·58641		
0·9	0·86047	5·9	1·43018	10·9	1·58743	15·9	1·63280	20·9	1·59077		
1·0	0·94608	6·0	1·42469	11·0	1·57831	16·0	1·63130	21·0	1·59489		
1·1	1·02869	6·1	1·42087	11·1	1·56927	16·1	1·62291	21·1	1·59873		
1·2	1·10805	6·2	1·41871	11·2	1·56042	16·2	1·62657	21·2	1·60225		
1·3	1·18396	6·3	1·41817	11·3	1·55182	16·3	1·62339	21·3	1·60543		
1·4	1·25623	6·4	1·41922	11·4	1·54356	16·4	1·61973	21·4	1·60823		
1·5	1·32468	6·5	1·42179	11·5	1·53571	16·5	1·61563	21·5	1·61063		
1·6	1·38918	6·6	1·42582	11·6	1·52835	16·6	1·61112	21·6	1·61261		
1·7	1·44959	6·7	1·43121	11·7	1·52155	16·7	1·60627	21·7	1·61415		
1·8	1·50582	6·8	1·43787	11·8	1·51535	16·8	1·60111	21·8	1·61525		
1·9	1·55778	6·9	1·44570	11·9	1·50981	16·9	1·59572	21·9	1·61590		
2·0	1·60541	7·0	1·45460	12·0	1·50497	17·0	1·59014	22·0	1·61608		
2·1	1·64870	7·1	1·46443	12·1	1·50088	17·1	1·58443	22·1	1·61582		
2·2	1·68763	7·2	1·47509	12·2	1·49755	17·2	1·57865	22·2	1·61510		
2·3	1·72221	7·3	1·48644	12·3	1·49501	17·3	1·57285	22·3	1·61395		
2·4	1·75249	7·4	1·49834	12·4	1·49327	17·4	1·56711	22·4	1·61238		
2·5	1·77852	7·5	1·51068	12·5	1·49234	17·5	1·56146	22·5	1·61041		
2·6	1·80039	7·6	1·52331	12·6	1·49221	17·6	1·55598	22·6	1·60806		
2·7	1·81821	7·7	1·53611	12·7	1·49287	17·7	1·55070	22·7	1·60536		
2·8	1·83210	7·8	1·54894	12·8	1·49430	17·8	1·54568	22·8	1·60234		
2·9	1·84219	7·9	1·56167	12·9	1·49647	17·9	1·54097	22·9	1·59902		

3·0	1·84865	8·0	1·57419	13·0	1·49936	18·0	1·53661	23·0	1·59546		
3·1	1·85166	8·1	1·58637	13·1	1·50292	18·1	1·53264	23·1	1·59168		
3·2	1·85140	8·2	1·59810	13·2	1·50711	18·2	1·52909	23·2	1·58772		
3·3	1·84808	8·3	1·60928	13·3	1·51188	18·3	1·52600	23·3	1·58363		
3·4	1·84191	8·4	1·61981	13·4	1·51716	18·4	1·52339	23·4	1·57945		
3·5	1·83313	8·5	1·62960	13·5	1·52291	18·5	1·52128	23·5	1·57521		
3·6	1·82195	8·6	1·63857	13·6	1·52905	18·6	1·51969	23·6	1·57097		
3·7	1·80862	8·7	1·64665	13·7	1·53352	18·7	1·51863	23·7	1·56676		
3·8	1·79339	8·8	1·65379	13·8	1·54225	18·8	1·51810	23·8	1·56262		
3·9	1·77650	8·9	1·65993	13·9	1·54917	18·9	1·51810	23·9	1·55860		
4·0	1·75820	9·0	1·66504	14·0	1·55621	19·0	1·51863	24·0	1·55474		
4·1	1·73874	9·1	1·66908	14·1	1·56330	19·1	1·51967	24·1	1·55107		
4·2	1·71837	9·2	1·67205	14·2	1·57036	19·2	1·52122	24·2	1·54762		
4·3	1·69732	9·3	1·67393	14·3	1·57733	19·3	1·52324	24·3	1·54444		
4·4	1·67583	9·4	1·67473	14·4	1·58414	19·4	1·52572	24·4	1·54154		
4·5	1·65414	9·5	1·67446	14·5	1·59072	19·5	1·52863	24·5	1·53897		
4·6	1·63246	9·6	1·67316	14·6	1·59702	19·6	1·53192	24·6	1·53672		
4·7	1·61101	9·7	1·67084	14·7	1·60296	19·7	1·53557	24·7	1·53484		
4·8	1·58998	9·8	1·66757	14·8	1·60851	19·8	1·53954	24·8	1·53333		
4·9	1·56956	9·9	1·66338	14·9	1·61360	19·9	1·54378	24·9	1·53221	50·0	1·55162

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TABLE C

$F(\alpha) = -\frac{1}{2}\cos\alpha\, S_1(2\alpha) + \frac{1}{2}\sin\alpha\, S_2(2\alpha)$ $\qquad$ $G(\alpha) = \frac{1}{2}\sin\alpha\, S_1(2\alpha) + \frac{1}{2}\cos\alpha\, S_2(2\alpha)$

α	F (α)	G (α)	α	F (α)	G (α)	α	F (α)	G (α)	α	(F α)	G(α)
0·1	0·00499	0·09963	2·1	1·292	0·5076	4·1	0·08349	−1·509	6·1	−1·667	0·4509
0·2	0·01993	0·1985	2·2	1·339	0·4153	4·2	−0·06858	−1·530	6·2	−1·614	0·6149
0·3	0·04459	0·2943	2·3	1·375	0·3136	4·3	−0·2218	−1·536	6·3	−1·545	0·7723
0·4	0·07893	0·3874	2·4	1·401	0·2052	4·4	−0·3758	−1·529	6·4	−1·460	0·9234
0·5	0·1214	0·4726	2·5	1·416	0·09074	4·5	−0·5273	−1·504	6·5	−1·360	1·067
0·6	0·1734	0·5540	2·6	1·420	−0·03139	4·6	−0·6753	−1·466	6·6	−1·248	1·200
0·7	0·2317	0·6261	2·7	1·410	−0·1536	4·7	−0·8196	−1·413	6·7	−1·120	1·326
0·8	0·2978	0·6904	2·8	1·388	−0·2790	4·8	−0·9574	−1·345	6·8	−0·9817	1·437
0·9	0·3698	0·7456	2·9	1·354	−0·4066	4·9	−1·089	−1·263	6·9	−0·8328	1·536
1·0	0·4465	0·7902	3·0	1·307	−0·5333	5·0	−1·210	−1·168	7·0	−0·6757	1·621
1·1	0·5273	0·8239	3·1	1·247	−0·6594	5·1	−1·321	−1·059	7·1	−0·5089	1·690
1·2	0·6102	0·8463	3·2	1·175	−0·7794	5·2	−1·422	−0·9397	7·2	0·3377	1·743
1·3	0·6966	0·8572	3·3	1·091	−0·8962	5·3	−1·509	−0·8095	7·3	−0·1614	1·779
1·4	0·7820	0·8555	3·4	0·9951	−1·013	5·4	−1·583	−0·6694	7·4	+0·01726	1·798
1·5	0·8672	0·8414	3·5	0·8902	−1·110	5·5	−1·646	−0·5217	7·5	0·1976	1·800
1·6	0·9491	0·8160	3·6	0·7745	−1·205	5·6	−1·688	−0·3679	7·6	0·3769	1·783
1·7	1·030	0·7765	3·7	0·6495	−1·289	5·7	−1·716	−0·2077	7·7	0·5534	1·750
1·8	1·105	0·7257	3·8	0·5157	−1·361	5·8	−1·729	−0·04376	7·8	0·7264	1·698
1·9	1·175	0·6639	3·9	0·3779	−1·424	5·9	−1·725	+0·1216	7·9	0·8930	1·630
2·0	1·237	0·5916	4·0	0·2324	−1·473	6·0	−1·704	0·2893	8·0	1·052	1·545

The above table is based on Professor P. O. PEDERSEN's Tables A and B of the functions $S_1(\alpha)$ and $S_2(\alpha)$, making use of the formulae (32) of this paper. The calculations were carried out by four-figure logarithm tables, so that, although four significant figures are given, the last figure may be in error by two or three units.

TABLE D

$R(\alpha) = F(2\alpha) - 4\cos\alpha\, F(\alpha)$ $\qquad$ $U(\alpha) = \sin\alpha\, L(\alpha) - \cos\alpha\, R(\alpha)$

$L(\alpha) = G(2\alpha) - 4\cos\alpha\, G(\alpha)$ $\qquad$ $V(\alpha) = \cos\alpha\, L(\alpha) + \sin\alpha\, R(\alpha)$

α	U (α)	V (α)	R (α)	L (α)	α	U (α)	V (α)	R (α)	L (α)
0·2	−0·07823	−0·3832	0·000527	−0·3912	2·2	1·175	2·567	2·767	−0·5606
0·4	−0·2930	−0·6700	0·00826	−0·7281	2·4	1·842	2·689	3·174	−0·7379
0·6	−0·5860	−0·784	0·0401	−0·9763	2·6	2·406	2·669	3·438	−1·047
0·8	−0·8788	−0·687	0·1195	−1·109	2·8	2·865	2·525	3·546	−1·419
1·0	−1·0840	−0·375	0·2701	−1·114	3·0	2·178	2·294	3·470	−1·822
1·2	−1·1393	+0·1101	0·5153	−1·022	3·2	3·355	1·998	3·232	−2·192
1·4	−0·9935	0·698	0·8577	−0·8600	3·4	3·404	1·646	2·870	−2·460
1·6	−0·6464	1·307	1·288	−0·6844	3·6	3·330	1·230	2·440	−2·576
1·8	−0·1268	1·855	1·778	−0·5446	3·8	3·132	0·771	2·004	−2·525
2·0	+0·499	2·296	2·295	−0·5018	4·0	2·829	0·252	1·655	−2·301

The above table was computed from formulae (38), making use of Professor P. O. PEDERSEN's Tables A and B of the functions $S_1(\alpha)$ and $S_2(\alpha)$. As four-figure logarithms were employed, the last significant figure may be in error by two or three units.

TABLE E—RADIATION RESISTANCE AND REACTANCE

α	$\alpha°$	l/λ	m'	M″	s'	S″	c'	C″	$\frac{C(\frac{1}{2}\pi)}{1-\cos\alpha}$	$\frac{S(\frac{1}{2}\pi)}{1-\cos\alpha}$
	° ′									
0·2	11–27	0·0318	2·0067	0·199	−0·391	0·0005	−0·0393	0·0062	−2·990	0·00502
0·4	22–55	0·0637	2·0255	0·398	−0·731	0·0077	−0·1495	0·0411	−2·938	0·01267
0·6	34–22	0·0955	2·0594	0·594	−0·975	0·0393	−0·3061	0·130	−2·856	0·03436
0·8	45–30	0·1273	2·1050	0·786	−1·104	0·1191	−0·4708	0·282	−2·753	0·1055
1·0	57–18	0·1591	2·1626	0·973	−1·088	0·267	−0·6262	0·492	−2·642	0·1957
1·2	68–45	0·1910	2·2316	1·154	−0·948	0·494	−0·7278	0·745	−2·537	0·3200
1·4	80–13	0·2228	2·3112	1·326	−0·671	0·812	−0·7522	1·009	−2·432	0·4771
1·6	91–40	0·2546	2·4004	1·490	−0·375	1·200	−0·6956	1·253	−2·346	0·6600
1·8	103–08	0·2865	2·4984	1·648	−0·011	1·648	−0·5623	1·441	−2·282	0·8680
2·0	114–35	0·3183	2·6040	1·794	+0·382	2·098	−0·3790	1·565	−2·250	1·079
2·2	126–03	0·3501	2·7172	1·930	0·750	2·530	−0·146	1·581	−2·240	1·308
2·4	137–30	0·3820	2·8336	2·056	1·106	2·916	+0·114	1·517	−2·248	1·530
2·6	148–58	0·4138	2·9554	2·172	1·431	3·188	0·352	1·371	−2·278	1·731
2·8	160–26	0·4456	3·0800	2·276	1·712	3·353	0·576	1·153	−2·332	1·935
3·0	171–53	0·4774	3·2064	2·372	1·950	3·379	0·766	0·884	−2·382	2·112
3·2	183–21	0·5093	3·3576	2·454	2·140	3·272	0·922	0·581	−2·442	2·269
3·4	194–48	0·5411	3·4608	2·532	2·270	3·036	1·044	0·255	−2·484	2·412
3·6	206–16	0·5729	3·5962	2·592	2·306	2·677	1·128	−0·102	−2·510	2·550
3·8	217–43	0·6048	3·7100	2·642	2·289	2·200	1·169	−0·433	−2·519	2·677
4·0	229–11	0·6366	3·8306	2·688	2·170	1·638	1·173	−0·677	−2·474	2·809

TABLE F—RADIATION RESISTANCE AND REACTANCE FOR LOG $(2l/a) = 4\cdot5$, AND $h/l = 0\cdot01$

α	$\|M\|$	$m°$ (° ′)	$\|S\|$	$S°$ (° ′)	$w''(h)$	$w'(h)$	$w°$ (° ′)	$R_1(\alpha)$	$R_2(\alpha)$	$R(\alpha)$
0·2	6·997	1–38	0·3817	0–04	0·1714	−0·00075	90–15	0·8417	0·0278	0·000527
0·4	6·984	3–16	0·7859	0–34	0·3332	−0·00265	90–28	0·8369	0·0560	0·00826
0·6	6·968	4–54	1·221	1–51	0·4772	−0·00466	90–33	0·8286	0·0829	0·04009
0·8	6·939	6–30	1·682	3–43	0·5964	−0·00624	90–36	0·8178	0·1056	0·1195
1·0	6·905	8–06	2·182	7–02	0·6835	−0·00294	90–15	0·8053	0·1267	0·2701
1·2	6·866	9–41	2·686	10–36	0·7364	+0·00313	89–45	0·7923	0·1449	0·5153
1·4	6·821	11–13	3·207	14–40	0·7508	0·01415	88–55	0·7785	0·1595	0·8577
1·6	6·766	12–43	3·621	19–21	0·7337	0·03091	87–35	0·7650	0·1713	1·288
1·8	6·707	14–13	4·013	24–14	0·6794	0·05202	85–37	0·7523	0·1809	1·778
2·0	6·645	15–40	4·312	29–07	0·5937	0·07548	82–45	0·7415	0·1888	2·295
2·2	6·572	17–03	4·489	34–14	0·4823	0·1013	78–08	0·7313	0·1936	2·767
2·4	6·500	18–27	4·567	39–41	0·3481	0·1273	69–55	0·7220	0·1987	3·174
2·6	6·424	19–46	4·509	45–00	0·1974	0·1498	52–49	0·7142	0·2033	3·438
2·8	6·351	21–01	4·332	50–42	0·0387	0·1689	12–54	0·7082	0·2063	3·546
3·0	6·261	22–16	4·040	56–45	−0·1248	0·1827	−34–21	0·7011	0·2104	3·470
3·2	6·138	23–09	3·632	63–37	−0·2835	0·1920	−55–54	0·6954	0·2083	3·232
3·4	6·087	24–34	3·194	71–51	−0·4333	0·1928	−66–01	0·6870	0·2177	2·870
3·6	5·994	25–37	2·696	83–03	−0·5635	0·1895	−71–25	0·6778	0·2197	2·440
3·8	5·934	26–26	2·223	98–12	−0·6703	0·1778	−75–08	0·6701	0·2196	2·004
4·0	5·826	27–28	1·894	120–10	−0·7491	0·1624	−77–46	0·6540	0·2200	1·655

TABLE F—(continued)

α	R (h)/c	|C|	C° (° ′)	v′(h)	v″ (h)	v° (° ′)	X (h)/R (h)	R (h) ohms	X (h) ohms	|Z (h)| ohms
0·2	0·01272	3·895	0–05	1·2582	−0·01320	− 0–36	−67·41	0·38	− 25·72	25·72
0·4	0·05130	3·549	0–40	1·1745	−0·01743	− 0–51	−43·51	1·54	− 66·96	66·96
0·6	0·1221	3·008	2–29	1·0395	−0·02799	− 1–33	−27·27	3·66	− 99·93	100·0
0·8	0·2283	2·343	6–55	0·8610	−0·03917	− 2–36	−17·86	6·85	−122·5	122·7
1·0	0·3841	1·619	17–43	0·6458	−0·05107	− 4–31	−11·99	11·52	−138·2	138·7
1·2	0·6180	1·042	45–38	0·4057	−0·06243	− 8–45	− 6·691	18·54	−124·0	125·4
1·4	0·9590	1·012	93–58	0·1506	−0·07168	− 25–26	− 2·210	28·77	− 63·54	69·78
1·6	1·407	1·494	122–58	−0·1082	−0·07711	−144–32	+ 1·285	42·21	+ 56·67	71·73
1·8	2·294	2·062	135–39	−0·3602	−0·07714	−167–55	3·383	68·82	232·8	242·8
2·0	3·753	2·582	142–40	−0·5966	−0·07174	−173–07	3·972	112·59	447·0	461·1
2·2	6·730	2·964	147–46	−0·7061	−0·05906	−175–13	3·344	201·9	675·3	714·7
2·4	12·95	3·225	151–56	−0·9764	−0·04146	−177–34	2·412	388·5	937·5	1014·9
2·6	30·83	3·376	156–02	−1·1192	−0·01927	−179–01	1·273	924·9	1176·9	1496·7
2·8	64·30	3·409	160–14	−1·2106	+0·00585	−180–17	0·234	1929·0	451·8	1981·2
3·0	37·98	3·329	164–35	−1·2538	0·03169	−181–27	− 0·647	1139·4	−737·4	1357·2
3·2	14·53	3·139	169–20	−1·2479	0·05527	−182–32	− 1·353	435·9	−589·5	733·2
3·4	6·625	2·849	174–52	−1·1876	0·07403	−183–34	− 1·915	198·7	−381·0	429·6
3·6	3·504	2·475	177–38	−1·0863	0·08145	−184–17	− 2·386	105·1	−250·9	270·5
3·8	2·073	2·054	192–10	−0·9262	0·1080	−186–39	− 2·536	62·19	−157·8	169·6
4·0	1·342	1·602	205–01	−0·7373	0·09283	−187–11	− 2·837	40·26	−114·2	121·0

TABLE G—RADIATION RESISTANCE AND REACTANCE FOR LOG $(2l/a) = 10\cdot58$ AND $h/l = 0\cdot01$

α	$\|M\|$	$m°$	$\|S\|$	S°	$w''(h)$	$w'(h)$	$w°$	$R_1(\alpha)$	$R_2(\alpha)$	$\frac{30\,R(\alpha)}{\sin^2\alpha}$
		° ′		°			° ′			
0·2	19·15	0–35	1·590	0–01	0·1572	−0·00041	90–09	0·7836	0·0101	0·400
0·4	19·14	1–11	3·154	0–09	0·3070	−0·00149	90–17	0·7819	0·0203	1·63
0·6	19·11	1–47	4·654	0–29	0·4429	−0·00276	90–32	0·7791	0·0302	3·78
0·8	19·07	2–22	6·042	1–02	0·5591	−0·00368	90–22	0·7757	0·0382	6·95
1·0	19·03	2–56	7·291	2–06	0·6499	−0·00279	90–15	0·7717	0·0461	11·4
1·2	18·97	3–29	8·326	3–24	0·7125	−0·00032	90–01	0·7677	0·0523	17·8
1·4	18·90	4–02	9·133	5–06	0·7440	+0·00242	89–39	0·7636	0·0576	26·5
1·6	18·81	4–33	9·559	7–13	0·7459	0·01182	89–06	0·7593	0·0619	38·7
1·8	18·74	5–03	9·725	9–45	0·7152	0·02127	88–18	0·7560	0·0649	56·3
2·0	18·65	5–31	9·535	12–42	0·6555	0·03198	87–12	0·7534	0·0672	83·4
2·2	18·54	5–58	8·991	16–20	0·5699	0·04294	85–41	0·7508	0·0687	127·1
2·4	18·45	6–24	8·164	20–55	0·4612	0·05547	83–09	0·7491	0·0700	208·5
2·6	18·33	6–48	7·106	26–40	0·3330	0·06588	78–49	0·7475	0·0712	388·5
2·8	18·27	7–10	5·839	35–03	0·1934	0·07494	68–49	0·7469	0·0716	948·0
3·0	18·10	7–31	4·565	47–42	0·0447	0·08137	28–47	0·7457	0·0725	5230·0
3·2	17·96	7–44	3·509	68–49	−0·1057	0·08549	−51–02	0·7449	0·0714	28400·0
3·4	17·88	8–08	3·091	100–26	−0·2519	0·08638	−71–05	0·7440	0·0741	1320·0
3·6	17·75	8–24	3·574	131–30	−0·3875	0·08435	−77–43	0·7424	0·0743	374·0
3·8	17·64	8–37	4·598	151–25	−0·5079	0·07879	−81–11	0·7406	0·0739	160·2
4·0	17·53	8–49	5·793	163–34	−0·6073	0·07049	−83–23	0·7371	0·0731	86·8

TABLE G—(continued)

α	$\frac{R(h)}{c}$	|C|	$C°$	$v'(h)$	$v''(h)$	$v°$	$\frac{X(h)}{R(h)}$	R (h) ohms	X (h) ohms	|Z (h)| ohms
			° ′			° ′				
0·2	0·01431	2·065	179–50	0·9262	0·000392	0–01	— 429·7	0·393	— 168·8	168·8
0·4	0·05365	2·053	178–51	0·8674	0·0000468	0–00	— 202 2	1·609	— 325·5	325·5
0·6	0·1243	2·016	176–18	0·7726	—0·001764	— 0–08	— 114·6	3·729	— 426·9	426·9
0·8	0·2304	1·932	171–36	0·6464	—0·00532	— 0–28	— 68·74	6·912	— 475·2	475·2
1·0	0·3820	1·810	164–15	0·4939	—0·01055	— 1–13	— 39·05	11·46	— 447·6	447·7
1·2	0·6012	1·654	153–15	0·3223	—0·01723	— 3–04	— 18·57	18·04	— 334·8	335·3
1·4	0·9076	1·495	137–33	0·1390	—0·02459	— 10–02	— 5·86	27·23	— 159·5	161·8
1·6	1·343	1·405	116–53	—0·0487	—0·03185	—146–49	+ 1·424	40·29	+ 57·39	70·1
1·8	1·996	1·444	93–41	—0·2425	—0·03811	—171–04	5·326	59·88	319·5	325·1
2·0	3·049	1·639	72–55	—0·4072	—0·04301	—173–59	6·446	91·47	589·5	600·0
2·2	4·814	1·862	58–55	—0·5648	—0·04428	—175–31	6·446	144·4	930·9	942·1
2·4	8·318	2·232	42–48	—0·6977	—0·04581	—176–14	5·334	249·5	1331·1	1354·0
2·6	16·83	2·525	32–53	—0·8039	—0·04397	—176–52	3·918	504·9	1997·6	2041·0
2·8	46·39	2·776	24–33	—0·8775	—0·04002	—177–23	2·268	1391·7	3151·0	3444·0
3·0	226·0	2·947	17–27	—0·9162	—0·03436	—177–51	0·5014	6780·0	3399·0	7584·0
3·2	97·95	3·042	11–01	—0·9180	—0·02723	—178–18	— 1·314	2938·5	—3861·0	4853·0
3·4	22·62	3·053	4–47	—0·8836	—0·01903	—178–46	— 3·136	678·6	—2128·8	2235·0
3·6	8·638	2·984	1–58	—0·8141	—0·01512	—178–56	— 5·043	259·1	—1306·5	1332·0
3·8	4·206	2·837	—8–47	—0·7106	+0·000234	—180–00	— 6·446	126·2	— 813·3	823·0
4·0	2·430	2·613	—15–01	—0·5975	0·00805	—180–48	— 7·683	72·9	— 559·8	564·5

SUMMARY OF FORMULAE FOR CALCULATION OF RADIATION RESISTANCE AND REACTANCE

l = height of antenna above a perfectly conducting earth.

a = radius of cylindrical antenna in same units as l.

λ = wave-length in same units as l and a.

h = height above perfect earth at which current and voltage are measured (may be identified with height of connexion to feeder).

ε = distance of lower end of antenna (supposed insulated) from perfect earth. The formulae given below suppose that $l \gg h \gtrapprox \varepsilon \gg a$.*

$\alpha = 2\pi l/\lambda$, sometimes expressed in "electrical degrees" ($l/\lambda = 1$, $\alpha^\circ = 360^\circ$).

$$\mathrm{S}_1(\alpha) = \int_0^\alpha \frac{1-\cos u}{u}\,du, \qquad \mathrm{S}_2(\alpha) = \int_0^\alpha \frac{\sin u}{u}\,du.$$

(Formulae (24) and (26), Tables A and B.)

$$\left.\begin{array}{l} m' = 2\{\mathrm{S}_1(\alpha) + \sin\alpha/\alpha\}, \qquad \mathrm{M}' = 2\log(2l/a) - m', \\ \mathrm{M}'' = 2\{\mathrm{S}_2(\alpha) - (1-\cos\alpha)/\alpha\} \end{array}\right\} \quad . \quad . \quad \text{(i)}$$

V (α), U (α)—*see* formulae (35) and (36) and Table D.

Radiation Resistance

$$\left.\begin{array}{l} s' = \mathrm{V}(\alpha) - \sin\alpha\,\mathrm{S}_1(2\alpha), \qquad \mathrm{S}'' = \mathrm{U}(\alpha) + \sin\alpha\,\mathrm{S}_2(2\alpha) \\ \mathrm{S}' = \sin\alpha\log(l/a) + 2\cos\alpha\sin(\alpha h/l)\,.\,\log(l/h) + s' \\ \tan s = \mathrm{S}''/\mathrm{S}' \qquad |\mathrm{S}| = (\mathrm{S}'^2 + \mathrm{S}''^2)^{\frac{1}{2}} = \mathrm{S}'\sec s \end{array}\right\} \quad . \quad . \quad \text{(ii)}$$

$w(h) = \mathrm{A}\{w'(h) - i\,w''(h)\}$ = current at height h above perfect earth.

A = real constant (current-amplitude constant expressed in absolute e.m. units).

$w'(h)$, $w''(h)$ components of current in phase quadrature.

$$\left.\begin{array}{l} w'(h) = \tfrac{1}{2}|\mathrm{S}|/|\mathrm{M}|\sin(s-m)\,,\ w''(h) = \sin\alpha - \tfrac{1}{2}|\mathrm{S}|/|\mathrm{M}|\cos(s-m) \\ \tan w = w''(h)/w'(h)\,,\ |w(h)| = \{w'^2(h) + w''^2(h)\}^{\frac{1}{2}} \\ = w'(h)\sec w \end{array}\right\} \quad . \quad \text{(iii)}$$

* The case $\varepsilon = 0$ and the lower end of the antenna is in direct electrical contact with the perfectly conducting earth is not considered in this paper. Characteristics and methods of excitation of a grounded antenna are described in a recent paper by MORRISON and SMITH (1936). Such an antenna may be referred to as a shunt excited grounded antenna, the theory of which it is hoped to consider in a future paper.

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$$\left.\begin{aligned} C\left(\tfrac{1}{2}\pi\right) &= -S_1(2\alpha) + \cos\alpha\,\{2S_1(\alpha) + S_1(2\alpha)\} + U(\alpha) \\ S\left(\tfrac{1}{2}\pi\right) &= S_2(2\alpha) - \cos\alpha\,\{2S_2(\alpha) + S_2(2\alpha)\} - V(\alpha) \\ R_1(\alpha) &= \cos m - \frac{1}{2\,|M|}\left\{\log(4l/a) + \frac{C\left(\frac{1}{2}\pi\right)}{1-\cos\alpha}\right\} \\ R_2(\alpha) &= \sin m - \frac{1}{2\,|M|}\,\frac{S\left(\frac{1}{2}\pi\right)}{1-\cos\alpha} \end{aligned}\right\} \quad \text{(iv)}$$

$R(h)$ = radiation resistance referred to current measurements at height h above perfect earth, expressed in absolute e.m. units (cm./sec.).

$$\frac{R(h)}{c} = \left[\frac{1}{|w(h)|^2}\{R^2{}_1(\alpha) + R_2{}^2(\alpha)\}\,R(\alpha) + \int_0^{\alpha}\frac{f(\zeta)}{\zeta}\,d\zeta\right]. \quad \text{(v)}$$

$R(\alpha) = F(2\alpha) - 4\cos\alpha\,F(\alpha)$ (Formula (35) and Table D.)

c = ratio of electrical units $= 3 \times 10^{10}$ cm./sec. To express $R(h)$ in ohms we have $R(h)$ ohms $= 30\,\{R(h)/c\}$. The exact calculation of radiation resistance requires the computation of the polar radiation diagram in each case. The integration of the Poynting vector over a distant hemisphere cannot be expressed in terms of tabulated functions and recourse must be had to quadratures. In the range $0 < \alpha < 4 \cdot 0$, $f(\theta)$ is computed from (77), and when expressed in terms of the variable $\zeta = \alpha \sin \frac{1}{2}\theta$ the integral in (v) is easily evaluated by graphical methods. At $\alpha = 3 \cdot 64$ it is a small fraction of the first term, $-5 \cdot 8\%$ for $\log(2l/a) = 4 \cdot 5$ and $-1 \cdot 3\%$ for $\log(2l/a) = 10 \cdot 58$, and of negative sign in both cases. For smaller values of α the value of the integral is, roughly, proportionally less, so that its omission in computing Tables F and G is not serious in the range $0 < \alpha < 4 \cdot 0$.

Radiation Reactance

$$\left.\begin{aligned} C' &= U(\alpha) + \cos\alpha\,S_1(2\alpha) \qquad C'' = V(\alpha) + \cos\alpha\,S_2(2\alpha) \\ C' &= \cos\alpha\,\{2\log(\tfrac{1}{2}l/h) - \log(l/a)\} + c' \\ &\tan c = C''/C', \quad |C| = (C'^2 + C''^2)^{\frac{1}{2}} = C' \sec c \end{aligned}\right\} \quad \text{(vi)}$$

$\sigma(h)$ = charge per unit length of antenna at height h from perfect earth measured in absolute e.s. units.

$$\left.\begin{aligned} -\sigma(h) &= A\,\{v'(h) - iv''(h)\} \qquad \tan v = v''(h)/v'(h) \\ v'(h) &= \cos\alpha + \tfrac{1}{2}\frac{|C|}{|M|}\cos(C+m)\,, \quad v''(h) = -\tfrac{1}{2}\frac{|C|}{|M|}\sin(C+m) \end{aligned}\right\} \quad \text{(vii)}$$

$X(h)$ = radiation reactance referred to current and voltage measurements at height h above perfect earth, expressed in same units as $R(h)$

$$X(h) = R(h)\tan(w - v) \quad \text{(viii)}$$

radiation impedance

$$= |Z(h)| = \{R^2(h) + X^2(h)\}^{\frac{1}{2}} = R(h)\sec(w - v). \quad \text{(ix)}$$

For convenience in computation these functions of α entering into the above formulae independent of antennae dimensions are entered in Table E, while Tables F and G refer to cylindrical antennae for which $\log(2l/a) = 4\cdot5$ and $10\cdot58$ respectively.

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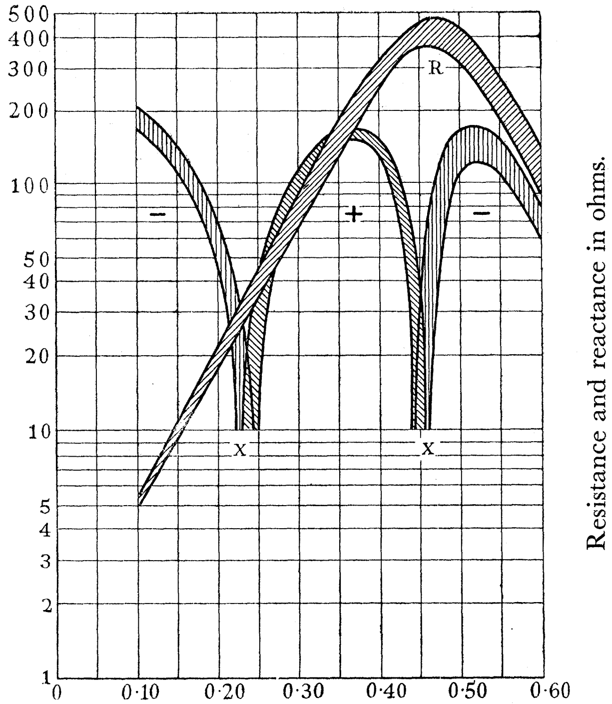

Physical height-fraction of a wave-length.

Fig. (i.)

Fig. (i) reproduces from a paper by Messrs. Chamberlain and Lodge references to base insulated doubly-tapered antennae towers. The general trend of the impedance characteristics is seen to follow roughly the theoretical diagrams for the perfectly conducting antenna. Recently excellent diagrams referring to a base insulated, series excited antenna tower of uniform square cross-section of 1/62 ft. side and 400 ft. long have been described in a paper by Messrs. Morrison and Smith of the Bell Telephone Laboratories.

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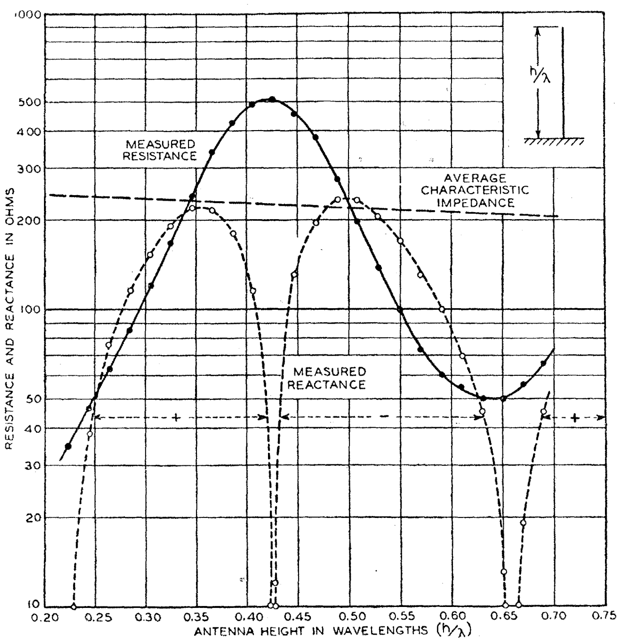

Fig. (ii)

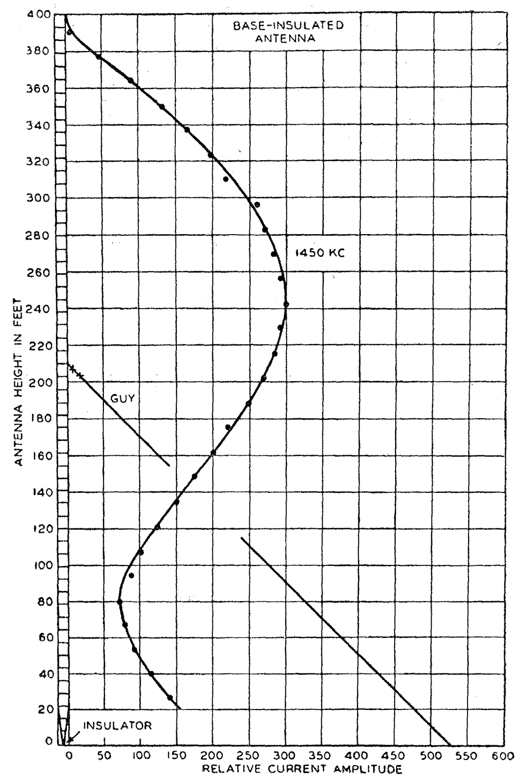

Fig. (iii)

With the kind permission of the Bell Telephone Company, the authors, and the Institute of Radio Engineers, figs. (ii) and (iii) are reproduced from this paper. It will be noted that there is again a general agreement with theory in the characteristics. It is evident that the effect of eddy currents in the earth near the base of the antenna has a marked influence on the radiation characteristics which depend on the current and density of electric charge near the antenna base at the point where current and " voltage " are measured. As mentioned in § 1, the method of this paper may be readily applied to evaluating the correction due to earth conductivity and non-uniform cross-section.